# THEORETICAL AND PRACTICAL GRAPHICS

AN EDUCATIONAL COURSE

ON THE

THEORY AND PRACTICAL APPLICATIONS

OF

# DESCRIPTIVE GEOMETRY AND MECHANICAL DRAWING

PREPARED FOR STUDENTS IN GENERAL SCIENCE, ENGINEERING OR ARCHITECTURE

BY

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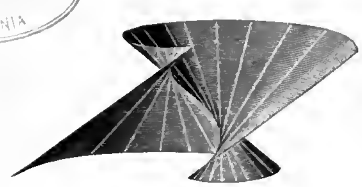

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## GROUPINGS OF CHAPTERS FOR INDEPENDENT COURSES.



I. Course in Free-Hand Sketching and Lettering, Note-Taking, Dimensioning and the Conventional Representation of Materials. Chapters II and VII.

II. The Choice and Use of Instruments; Line and Brush Work; Plane Problems of the Line and Circle; Projections (Third Angle Method); Development of Surfaces for Sheet Metal Constructions; Intersections; Working Drawings of Rail Sections, Bridge Post Details, Gearing, Springs, Screws, Bolts, Slide Valve, etc. Chapters III, IV, VI, X (to Art. 445), XVII and Appendix.

III. Course on the Helix, Conic Sections, Trochoidal Curves, Link-Motion Curves, Centroids, Spirals, etc. Chapter V and Appendix.

IV. Working Drawings by the Third Angle Method; Intersections and the Development of Surfaces. Chapter X to Art. 445.

V. Descriptive Geometry (Monge's), First Angle Method. Chapters IX, X (Arts. 445-522.)

VI. Shades, Shadows and Perspective, with especial reference to Architectural Applications. Chapters XIII and XIV.

VII. Axonometric Projection, Isometric Projection, One-plane Descriptive, Oblique (Clinographic) Projection, Cavalier Perspective. Chapters XV and XVI.

VIII. Broad Course in Descriptive Geometry, and its applications in Trihedrals, Spherical Projections, Shadows, Perspective, Axonometric and Oblique Projections. Chapters I, IX-XVI.





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# PREFACE

THE preparation of this work was not undertaken until the author had felt the need of such a book for his own classes, and a careful examination of the literature of graphical science had led to the conviction that it would occupy a distinct field.

So great had been the cost and so highly specialized the nature of the finer text-books on the topics here treated, that to give a broad, educational course, by using the best work available on each branch, involved a far greater outlay than the average student could well afford, or a teacher would feel justified in requiring him to make. Part of the self-imposed task, therefore, was to endeavor to compress between the covers of a book not larger than the average more specialized work, and at no greater cost to the student, not only all the usual matter found in treatises on mechanical drawing and orthographic projection, but also much which should—but too often does not—form a part of a draughtsman's education.

Of scarcely less importance than the proposed extended range of content was the method of presentation, the desire being not only to lay a broad and thorough foundation for advanced work along mathematico-graphical lines, but also in so doing to have every feature—illustrations, typography and even the quality of the paper—contribute as much as possible to the creation and increase of an interest in some of the topics for their own sake, and to a desire to continue to work in some of the fields into which the student would be here introduced.

While aiming to include nothing which might not reasonably be required of every candidate for a scientific degree, it was felt that it would increase the serviceability of the book, alike to teachers and to those dependent upon self-instruction, if it were so arranged that by taking its chapters in certain indicated groupings,* either elementary or advanced graphical courses could be taken from it with equal facility.

On its practical side it will be found in fullest accord with the modern methods of the leading engineering and architectural draughting offices. The Third Angle Method for making machine-shop drawings receives special consideration, independently of the earlier system; the latter, however, is of too great convenience for pure mathematical work and for stereotomy to ever become obsolete, and is therefore fully treated by itself.

Since but little new matter is presented, whatever especial value the book may be found to possess must in chief measure depend upon the way in which old facts are here stated, illustrated and correlated; but the following may, however, be mentioned as original, although previously issued either in pamphlet form or in the advance sheets which have for some time been in use with the author's classes: A method for drawing a tangent to a Spiral of Archimedes at a given point, when the pole and a portion only of the arc are given; a demonstration of the property of double generation of trochoidal curves when the tracing point is not on the circumference of the generator, with new terms completing a nomenclature of trochoids based on the property just mentioned; a simple

*See opposite page. Some of these groupings are also to be separately issued as "parts."

method for projecting the Plücker conoid; and a few new terms in Chapter IX, suggested in the interest of brevity.

The conchoidal hyperboloid of Catalan is probably treated in English for the first time, in this work; while such topics as the preparation of drawings for illustration, projective conics, relief perspective, the theory of centroids and certain of the higher plane curves and algebraic surfaces, are among the features which will be noted as unusual in an elementary treatise.

*The Title.* The comprehensive term *Graphics* was selected in the interest of brevity as well as appropriateness, as permitting the introduction of any science based upon the exact delineation of relations on paper, usually by the application of geometrical—and, in particular, of projective—properties by means of draughting instruments.

No rigid line can be drawn between the *theoretical* and the *practical* part, except as the grouping of the chapters, already alluded to, separates the elementary—and usually called "practical"—portions from the advanced; but a knowledge of the mathematical properties of the hyperbolic paraboloid, and the ability to make the drawings for a bridge portal of that form* when occasion requires, is obviously as "practical" as the drawing of an elbow joint; the classes these constructions represent therefore receive equal treatment, as this book is partly intended to be a concrete protest both against that spirit which regards a mathematical abstraction as degraded if some commercial application of it can be found, and against the disparagement of theory, as worthless for the "practical man."

*Chapter I.* A broad and comprehensive survey of the fields the student is about to enter seems the natural preliminary to intelligent work therein; the first chapter is, therefore, devoted to rigid definition and differentiation of the graphical sciences, and the arts in which they are applied. Some remarks on the nomenclature of geometries are also included, as further extending the draughtsman's usually too limited horizon.

This would naturally be followed by the ninth and succeeding chapters in a course arranged more for educational than commercial purposes.

*Chapter II.* As free-hand sketches are rightly made the basis of much of the practical draughting of the embryo engineer or architect, and as the graduate has frequent occasion, either as inspector or designer, to make clear and intelligible drawings without instruments, full instructions are given in this section as to what may be called technical, as distinguished from artistic, free-hand work, covering the following points: Sketching either in pictorial or orthographic view, dimensioning, free-hand lettering, conventional representation of materials, and note-taking on bridges and other trussed work, pins, bolts, screws, nuts and gearing.

*Chapters III and IV* are devoted to the description of the draughtsman's equipment, and to preliminary practice in its use, during which the student is familiarized with the methods of representation most employed, and with the solutions of the usual problems of the straight line and circle. The hyperboloid and anchor ring are also given as good tests of the beginner's skill in execution, but are so presented as to afford, with the other problems, material for recitation.

Since these chapters were electrotyped an instrument of exceptional value has been placed on the market, a compass whose legs remain parallel as the instrument opens. This is a novelty of such merit as to justify a notice here, since it cannot be incorporated in the body of the work.

*Chapter V*, although appearing at that stage of a beginner's work when he will presumably be learning the use of the irregular curve and being ostensibly to furnish exercises therefor, is in

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*Although an unusual design, one is in process of erection at present writing.

reality a treatise on the more important higher plane curves, and on the helix. It afforded an opportunity, in connection with the conic sections, to introduce the student to the beauties of the projective method, and give him his first notions of perspective.

The close analogy between homological plane and space figures made it seem advisable to introduce the latter, if at all, immediately after the former; so that relief-perspective appears somewhat out of its logical mathematical setting. While employing Cremona's notation, the works of Burmester, Wiener and Peschka have been otherwise followed on projective geometry.

The prominence given to the trochoidal curves, both in the main text and the Appendix, while primarily due to the interest in them which a reading of Proctor's *Geometry of Cycloids* aroused, is justified both by their intrinsic value, mathematically, and their important practical applications. Their tabular classification — an extension of Kennedy's scheme — contains distinctions among the hypo-curves whose acceptance by both Reuleaux and Proctor would seem to assure their permanence; while the reciprocal terms *Ortho-cycloid* and *Cyclo-orthoid*, incorporated at the suggestion of Professor Reuleaux, completed the system in a symmetrical manner.

The remaining plane curves are treated with varying degrees of fullness, according to the importance of their properties and applications; while throughout the chapter, as in other portions of the work, historical or descriptive matter has been introduced in order to enliven as far as possible what would otherwise have been a bare statement of mathematical fact.

Salmon, Leslie, Eagles and Proctor were the authorities of most service in this connection.

*Chapters VI and VII.* Proficiency with brush and colors is an indispensable qualification for success either as artist or architect. It is customary, however, in some quarters, to disparage such attainments in the engineer, as likely to be so infrequently in demand as to make the time spent in their acquisition a practical loss. If it is assumed that every student of engineering is to enter the draughting office of some bridge company, on the lowest round of the professional ladder, there to remain, ambitionless, then let him by all means learn only tracing and copying; but the instances of improved conditions, due to manual skill, are too numerous to justify any lowering of the standard for the embryo engineer, especially as he might otherwise find in later life, as has many another, a design that was inferior to his own accepted because more handsomely worked up. It is also well to remember, that in times of depression in the engineering world his abilities in this line and in lettering would aid him in other fields, and that superior skill in both, combined with originality, often commands the same rate per week in illustrating establishments as is paid per month for shop drawing. Chapters on the methods of obtaining varied effects, and on lettering, are therefore among the most important relating to the less theoretical part of the student's preparatory work.

The full instructions given in Chapter VII on spacing and proportioning, mechanical short-cuts, ornamentation, etc., will, it is believed, make this portion unusually serviceable to those who have felt the lack of such features in many otherwise most valuable works. In the Appendix a large number of complete alphabets affords a considerable range of choice, among forms which are of special service to engineers, architects and others.

*Chapter VIII.* In addition to acquaintance with the blue-print process, whose use is at present so well-nigh universal, some familiarity with other modern methods of graphic reproduction may well form a part of the education of a scientific man, both as a means of enhancing his interest in the work of others, and of enabling him, with the least expenditure of time, to prepare the drawings for the illustration of his own researches or original designs. Full information is therefore given in this chapter on all the technicalities with which it is requisite that the amateur illustrator should be familiar, and a list of reference works is furnished the intending specialist.

*Chapters IX and X.* In these chapters, covering an even hundred of pages, the Descriptive Geometry of Monge is treated in a manner intended not only to reduce to a minimum the difficulties ordinarily encountered in its study by students who are deficient in the imaginative faculty, but also at the same time to arouse an interest in this fundamental science of the constructive arts. Considerable reliance is placed, for the attainment of these ends, upon the use of pictorial views; and for the surfaces involved a series of wood-cuts are presented, which ought to prove a fair equivalent for a collection of models to those who unfortunately have not access to the latter.

Believing with Cremona that the association of the names of illustrious investigators with the products of their labors is "not without advantage in assisting the mind to retain the results themselves, and in exciting that scientific curiosity which so often contributes to enlarge our knowledge," the author has given both as to curves and surfaces, the commonly accredited source, although without undertaking verification.

The idea of defining a straight line as determined by two points (footnote to Art. 336) is due to Halsted (Appendix to translation of Bolyai), but since it was electrotyped it would seem to be an improvement to have it read "the line that is completely determined by any two of its points."

In Chapter X the choice is offered of dealing with figures by either the First Angle or Third Angle Methods. The latter is given first, being usually applied to more elementary surfaces than the other; and in connection with it the development of surfaces receives full treatment, followed by a large number of problems on the intersection of developable surfaces, which it is assumed will be worked out, like those in the section preceding them, to their logical conclusion—a finished model in Bristol-board.

Variations of the problems on projection, sections, etc., can be readily made by employing the designs given in the Appendix.

The portion of Chapter X which is devoted to the First Angle Method is supposed to be taken in close connection with Chapter IX, and may, if preferred, follow directly after a reading of pages 105–119, in order to model the course more closely along Continental lines.

*Chapter XI* is on Trihedrals, which are treated in the usual way, except that in several cases solutions are given by both the one-plane and two-plane methods.

*Chapter XII*, on Spherical Projections, differs from the usual treatment of the topic considerably, the scientific classification of Craig having been adopted, much of the space usually devoted to orthographic projection having been transferred to stereographic, and a larger number of methods described than in other elementary treatises on this topic.

*Chapters XIII and XIV*, on Shadows and Perspective, have been written with especial reference to the needs of architectural draughtsmen, and, though brief, cover all necessary principles, and the methods of best American practice.

*Chapters XV and XVI* give not only the theory of axonometric and oblique projections, but also their applications in shadows, timber framings and stone cutting; and the contrast between the two systems is shown more clearly by applying them to the same arch voussoirs and structural articulations. The method of drawing crystals in oblique projection is also illustrated.

One-plane Descriptive Geometry receives brief treatment, as being in theory so simple and in application so limited as to warrant the devotion to it of but little space.

*Chapter XVII*, on bridge details, gearing, screws, springs, etc., might more logically have followed the theory of the Third Angle Method in Chapter X, but would there have interrupted the continuity of that portion, and was therefore relegated to its present position. It is supplemented by working drawings in the Appendix.

*The Illustrations.* Believing that a good illustration reduces very materially the number of words necessary to a demonstration, the author has taken especial pains in designing and drawing the figures, so as to have them, in as large degree as possible, self-explanatory; and for their reproduction the five modern illustrative processes have been employed which seemed best adapted to the purpose, viz., cerography, photo-engraving, "half-tone," photo-gravure and wood-engraving. With regard to some of the figures the following acknowledgments are due:

The wood-cut of the Plücker conoid was made, by kind permission of Sir Robert Ball, from his illustration of that surface in his *Theory of Screws*, and is an exact reduction thereof, to scale.

Figures 90 and 91 are slight modifications of designs by Adhemar.

Figure 95 is from a photograph of a model by Burmester.

Figure 99 is in its essential features a combination of two illustrations in Reuleaux' *Kinematics*.

For the adaptation of the principle of the wedge to the tractrix (Fig. 115) indebtedness must be expressed to Halliday's *Mechanical Graphics*.

It is impossible to give credit for Figures 138 and 141, as their origin is unknown.

Figures 208, 211, 212 and 224–227 are from surfaces in Princeton's mathematical collection.

Figures 345, 346, 370 and 371 are half-tone reproductions of photographs taken at the Paris Conservatoire for Columbia University, a duplicate set of which were made for Princeton from the original negatives, which were kindly loaned the author for that purpose by the late Dr. F. A. P. Barnard, then president of Columbia.

*Reference Literature.* The more important treatises consulted are mentioned at the end of the book, as constituting a valuable reference library for the specialist in any of the lines named. The list includes some works already referred to in the text, as also those mentioned under some of the previous topical headings. There is so much in common in them that it has been impossible in many cases to say which has been an "original" source; but credit has been given whenever it could be with definiteness. Being the fortunate possessor of a copy of the first edition of Monge's Descriptive Geometry, there was at hand one authority, at least, whose originality was beyond doubt.

*With the following concluding remarks* a long and frequently interrupted undertaking is completed, and a foundation course in graphical science presented on a University plane, in such shape, it is hoped, as to be almost as serviceable to those who cannot use it amid University surroundings, as to the more fortunate ones who can. These remarks would include the conventional acknowledgments to advisers, proof-readers and publishers had not the original plan been adhered to, of having the work represent only so near an approach to an ideal then in mind as could be secured by carrying it through to a finished edition under the author's personal supervision of every feature.

Having purchased new type in order to have the plates flawless, and the final type-proofs having practically been such, it is a disappointment to find that standard unattained in the end; equally so to have a few of the later illustrations fall below the general average. Others represent the second or even third attempt of the plate-maker, notably Fig. 228 (b), which, however, as finally accepted, is a triumph of the engraver's art.

Previous editions of some of the earlier pages were printed from the type, for their care with which acknowledgment is due to the press-men, Messrs. J. P. Leigh and P. Bennett, of Princeton.

With the exception of a page of designs in the Appendix, material for the variation of problems is left for separate issue; as also chapters on valve motion, stereotomy and perspective of reflections.

F. N. W.

PRINCETON, N. J., JULY, 1897.

# TABLE OF CONTENTS

# THEORETICAL AND PRACTICAL GRAPHICS.

## *CHAPTER I.*

### FIRST PRINCIPLES, WITH GENERAL SURVEY OF THE FIELD OF GRAPHIC SCIENCE.

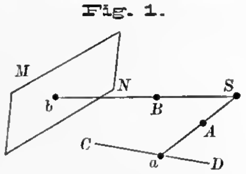
1. Geometrically considered, any combination of points, lines and surfaces is called a *figure.*

A figure lying wholly in one plane is called a *plane figure;* otherwise a *space figure.*

2. Among the methods of investigating and demonstrating the mathematical properties of figures, and of solving problems relating to them, that called *projection* is at once one of the most valuable and interesting, constituting, as it does, the common basis of nearly all graphic representations, whether of artist, architect or engineer.

When using this method figures are always considered in connection with a certain point called a *centre of projection.*

Fig. 1.




In Fig. 1 let $S$ be an assumed centre of projection and $A$ any point in space. The straight line $SA$, joining $S$ with $A$, is called a *projecting line* or *ray*, or simply a *projector*, and its intersection, $a$, with any line $CD$, is its *projection* upon that line. It is otherwise expressed by saying that $A$ is projected upon $CD$ at $a$.

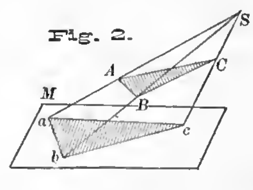
In the same way the point $B$ is projected[1] from $S$ upon the plane $MN$ at $b$; or, in other words, $b$ is—for the assumed position of $S$—the projection of $B$ upon the plane.

It is with projection upon a plane that we are principally concerned.

The word "projection" is used not only to indicate the method of representation but also the representation itself. In certain other branches of mathematics it has a yet more extended significance, being employed to denote the representation of any curve or surface upon any other.

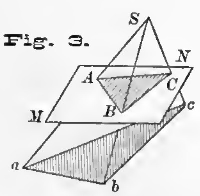
3. A figure, as $ABC$ (Fig. 2), is projected upon a plane, $MN$, by drawing projectors, $SA$, $SB$, $SC$, through its vertices and prolonging them, if necessary[2], to meet the plane. The figure $abc$, formed by joining the points in which the projectors intersect the plane, is then the projection of the first, or *original* figure.

Fig. 2.




The plane upon which the projection is made is called the *plane of projection.*

Fig. 3.




4. Were $abc$ (Fig. 3) the original figure and $MN$ the plane of projection, then would $ABC$ be the projection desired. Each figure may thus be considered a projection of the other for a given position of $S$, and when so related figures are said to *correspond* to each other. Points that are *collinear* (or *in line*) with the centre of projection, as $a$ and $A$, are called *corresponding points.*

[1] Were $S$ the muzzle of a gun, and $B$ a bullet speeding from it toward the plane, it would be projected against or through the plane at $b$. The appropriateness of the term "projection" is obvious.

[2] In Fig. 3 the projectors meet the plane between the centre $S$ and the given figure.

5. Having indicated what projections are and how obtained, it will be well, before giving their grand divisions and sub-divisions, to state the nature and extent of the field in which they may be employed.

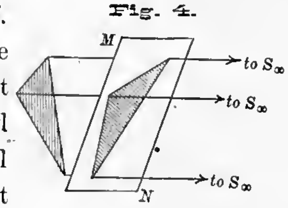
The mathematical properties of geometrical figures, as also the propositions and problems involving them, are divided into two classes, *metrical* and *descriptive*. In the first class the idea of *quantity* necessarily enters, either directly—as in measurement, or indirectly—as in ratio.[1] In the second or descriptive class, however, we find involved only those properties dependent upon *relative position*.[2] Descriptive properties are unaltered by projection, while, as ordinarily regarded, but few metrical properties are projective.[3] The main province of projection is obvious.

6. Descriptive Geometry is that branch of mathematics in which figures are represented and their descriptive properties investigated and demonstrated by means of projection.

### *DIVISIONS OF PROJECTION.*

7. All projections may be divided into two general classes, *Central* and *Parallel*.

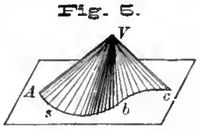
Fig. 4.




If the centre of projection be at a finite distance, as in Figs. 2 and 3, the projection obtained is called a *central projection*; but if we suppose it to be at infinity, as in Fig. 4, projectors from it will then evidently be parallel, and the resulting figure is called a *parallel projection* of the original figure. Parallel projection is thus seen to be merely a special case of central projection, yet each has been independently developed to a high degree and has an extensive literature.

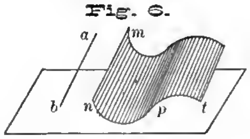
8. The terms *Conical* and *Cylindrical* are employed by many writers synonymously with central and parallel respectively. Central projections are also occasionally called *Radial* or *Polar*.

Fig. 5.




REMARK.—A straight line is said to generate a conical surface (see Fig. 5) when it constantly passes through a fixed point (the *vertex*), and is guided in its motion by a given fixed curve (the *directrix*). The moving straight line is a *generatrix* of the surface, and its various positions are called *elements* of the surface.

Fig. 6.




If the vertex of a conical surface be removed *to infinity* the elements will become parallel, and we shall have a *cylindrical* surface, which may be also defined as the surface (see Fig. 6) generated by a straight line that is guided in its motion by a given fixed curve, and is in any position parallel to a given, fixed, straight line.

The origin of the terms *conical* and *cylindrical* as applied to projection is obvious.

We have now to mention the more important sub-divisions of projections, with the sciences based upon them. The names depend in certain cases upon the nature of the centre of projection, while in others they are due to some particular application.

Under *Central* (or *Conical*) *projection* we have:—

9. *Projective Geometry* (*Geometry of Position*). While in its most general sense this science includes all projections, yet in its ordinary acceptation it may be defined as that branch of mathematics in which—*with the centre of projection considered as a mathematical point at a finite distance from the line or plane of projection*—the projective properties of figures are investigated and established.

[1] The following are *metrical* relations:
(a) The lateral area of a cylinder is equal to the product of the perimeter of its right section by an element of the surface.
(b) Two tetrahedrons which have a trihedral angle of the one equal to a trihedral angle of the other, are to each other as the products of the three edges of the equal trihedral angles.

[2] Illustrating *descriptive* or *positional* properties:
(a) If a line is perpendicular to a plane, any plane containing the line will also be perpendicular to the plane.
(b) Planes that are perpendicular to the same straight line are parallel to each other.

[3] See Klein's *Review of Recent Researches in Geometry* regarding the point of view which enables the projective method to include the whole of geometry.

Its chief practical application is in *Graphical Statics,* in which the stresses in bridge and roof trusses or other engineering constructions are determined graphically, by means of diagrams.

10. *Perspective.—If the centre of projection is the eye* of the observer the projection is called a *perspective* or *scenographic projection,* or—more commonly—simply a *perspective.* The plane of projection is then called the *perspective plane* or *picture plane,* and is always vertical. The position of the eye is called the *point of sight* or *station point,* and the projectors are termed *visual rays.*

Applied in the graphical construction usually preliminary to art work in water colors or oil; also in architectural perspectives and in scientific illustrations of machinery, etc.

It may be remarked that *any projection, central or parallel, presents to the eye the same appearance as the figure projected would if viewed from the centre of projection.*

11. *Relief-perspective.*—This differs from the perspective just defined in requiring, in addition to the usual perspective plane, a second plane parallel to it called a *vanishing plane,* the required representation appearing *in relief* between the two planes—a solid perspective, so to speak.

Employed chiefly in the construction of bas-reliefs and theatre decorations.

12. *Sciography* or *Shadows* (*artificial light*).—If the centre of projection is an artificial light, as the electric or that of a candle—either of which may, without appreciable error, be treated in graphical constructions as a mere point—the projectors will be *rays of light* and the projection will be the *shadow* of the figure projected.

Employed in obtaining shadow effects in paintings or architectural drawings.

13. *Photogrammetry* or *Photometrography,* the application of photography to surveying, the optical centre of the lens being the centre of projection.

14. Under *Parallel* (*or Cylindrical*) *projection* we have:—

(a) *Oblique* or *Clinographic,* and

(b) *Perpendicular* or *Orthographic,* also called *Orthogonal* or *Rectangular.*

These divisions are based upon the direction of the projectors with respect to the plane of projection, they being—as the names imply—inclined to it in oblique projection and perpendicular to it in orthographic.

### OBLIQUE PROJECTION.

15. *The shadow of an object in the sunlight* would be its oblique projection, the sun's rays being practically parallel.

16. Oblique projection is usually called *Clinographic* when employed in Crystallography.

17. In its other applications, when not simply called oblique, this projection is variously termed *Cavalier Perspective, Cabinet Projection* and *Military Perspective,* the plane of projection being vertical in the first and second, while in the last it is horizontal.

Oblique projection gives a pictorial effect closely analogous to a true perspective, yet is far more simple in its construction, and is much used for showing the form or method of assemblage of parts, or *details,* of machinery and architectural work; also in the representation of crystals.

### ORTHOGRAPHIC PROJECTION UPON A SINGLE PLANE.

18. When but one plane of projection is employed the only important applications of orthographic projection having special names are—

(a) *One-Plane Descriptive,* otherwise called *Horizontal Projection.*

Employed chiefly in fortification and general topographical work, in which the lines and surfaces represented are mainly horizontal.

(b) *Axonometric* (including *Isometric*) *Projection.*

Has the same range of application as oblique projection, viz., to objects whose lines lie mainly in directions mutually perpendicular to each other, or having axes so related.

ORTHOGRAPHIC PROJECTION UPON MUTUALLY PERPENDICULAR PLANES.

19. When upon two (or more) mutually perpendicular planes orthographic projection becomes the *Géométrie Descriptive* of Gaspard Monge, who reduced its principles to scientific form in the latter part of the eighteenth century.

The tendency — a logical one — toward the general adoption of the title "Descriptive Geometry" in the broad sense of Art. 6 would make it seem advisable to appropriate the name *Monge's Descriptive* to this — the most important division of graphic science, that we may not only find in it a hint as to its source but at the same time also pay to its inventor the honor of perpetual association of his name with his creation. As originally defined by Monge it is the application of orthographic projection (a) to the exact representation upon a plane surface, as that of a drawing-board, of all objects capable of rigorous definition, and (b) to the solution of problems relating to these objects in space and involving only their properties of form and position.

It might with propriety be divided into *pure* and *applied*, the former being the abstract science in which the mathematical relations existing between figures and their projections are examined and applied in the solution of certain fundamental problems of the point, line and plane; while the latter division would naturally include the application of these principles and methods to the solution of problems relating to the various elementary and higher mathematical surfaces, and to machine drawing and design, shades, shadows, perspective, stone-cutting, spherical projections, crystallography, pattern-making, carpentry, etc.

*ADDITIONAL REMARKS ON NOMENCLATURE.*

The student may find the following serviceable by way of enabling him to get clear ideas of the distinctions between certain divisions of geometrical science.

The term *Geometry*, unqualified, is usually understood to refer to the synthetic method of investigation of the form, position, ratio and measurement of geometrical figures, the reasoning being from particular to general truths by the aid of diagrams. In contra-distinction to other geometries it is frequently called *Euclidean*, after the celebrated Greek geometer, Euclid, (about 330–275 B. C.) who organized its theorems and problems into a science.

In *Coördinate* (or *Analytical*) Geometry the figure considered is referred to a system of coördinates, the relation between which, for every point of the figure, is expressed by means of an equation in which the coördinates are represented by algebraic symbols. The operations performed are algebraic, and the method of reasoning is from general to particular truths.

Although the invention of Analytical Geometry has been attributed to Descartes it is now recognized that he neither originated the use of coördinates nor the representation of curves and surfaces by means of equations. As the first to give complete scientific form to the analytic method his name has justly been given, however, to the most important division of coördinate geometry, *Cartesian*. But the writers of the present day under that head do not by any means confine themselves to the system of coördinates employed by him, which consisted of intersecting straight lines, usually perpendicular to each other.

In selecting the title "Projective Geometry" for the science defined in Art. 9 the eminent Cremona says, "I prefer not to adopt that of *Higher Geometry* (*Géométrie supérieure, höhere Geometrie*) because that to which the title 'higher' at one time seemed appropriate, may to-day have become very elementary; nor that of *Modern Geometry* (*neuere Geometrie*) which in like manner expresses a merely relative idea, and is moreover open to the objection that although the methods may be regarded as modern, yet the matter is to a great extent old. Nor does the title *Geometry of Position* (*Geometrie der Lage*) as used by Staudt seem to me a suitable one, since it excludes the consideration of the metrical properties of figures. I have chosen the name of *Projective Geometry* as expressing the true nature of the methods, which are based essentially upon central projection or perspective. And one reason which has determined this choice is that the great Poncelet, the chief creator of the modern methods, gave to his immortal book the title of *Traité des propriétés projectives des figures.*"

Cremona further states that "there is one important class of metrical properties (anharmonic properties) which are projective, and the discussion of which therefore finds a place in the Projective Geometry." But the positional definition given by Staudt for the anharmonic ratio of four points, which removes these properties from the class metrical to the class descriptive (which last are always projective), to that extent justifies the title employed by him, while making Cremona's choice none the less a fortunate one.

Among other geometries some belong to what may be called speculative mathematics, based upon "quasi-geometrical notions, those of more-than-three-dimensional space, and of non-Euclidean two-and-three-dimensional space, and also of the generalized notion of distance."*

The following will illustrate a method of arriving at a conception of non-Euclidean two-dimensional geometry. "Imagine the earth a perfectly smooth sphere. Understand by a plane the surface of the earth, and by a line the apparently straight line (in fact an arc of a great circle) drawn on the surface. What experience would in the first instance teach would be Euclidean two-dimensional geometry; there would be intersecting lines, which, produced a few miles or so, would seem to go on diverging, and apparently parallel lines which would exhibit no tendency to approach each other; and the inhabitants might very well conceive that they had by experience established the axiom that two straight lines cannot enclose a space, and also the axiom as to parallel lines. A more extended experience and more accurate measurements would teach them that the axioms were each of them false; and that any two lines, if produced far enough each way, would meet in two points: they would, in fact, arrive at a spherical geometry accurately representing the properties of the two-dimensional space of their experience. But their original Euclidean geometry would not the less be a true system; only it would apply to an ideal space, not the space of their experience."*

* Cayley.

## CHAPTER II.

ARTISTIC AND TECHNICAL FREE-HAND DRAWING.—SKETCHING FROM MEASUREMENT.—FREE-HAND LETTERING.—CONVENTIONAL REPRESENTATIONS.

20. Drawings, if classified as to the *method of their production*, are either *free-hand* or *mechanical;* while as to *purpose* they may be *working drawings*, so fully dimensioned that they can be worked from and what they represent may be manufactured; or *finished* drawings, illustrative or artistic in character and therefore shaded either with pen or brush, and having no hidden parts indicated by dotted lines as in the preceding division. Finished drawings also lack figured dimensions.

Working drawings of parts or "details" of a structure are called *detail drawings;* while the representation of a structure as a whole, with all its details in their proper relative position, hidden parts indicated by dotted lines, etc., is termed a *general* or *assembly* drawing.

21. While mechanical drawing is involved in making the various essential views—plans, elevations and sections—of all engineering and architectural constructions, and in solving the problems of *form* and *relative position* arising in their design, yet, to the engineer, the ability to sketch effectively and rapidly, *free-hand*, is of scarcely less importance than to handle the drawing instruments skillfully; while the success of an architect depends in still greater measure upon it.

We must distinguish, however, between *artistic* and *technical* free-hand work. The architect must be master of both; the engineer necessarily only of the latter.

To secure the adoption of his designs the architect relies largely upon the effective way in which he can finish, either with pen and ink or in water-colors, the perspectives of exterior and interior views; and such drawings are judged mainly from the artistic standpoint. While it is not the province of this treatise to instruct in such work a word of suggestion may properly be introduced for the student looking forward to architecture as a profession. He should procure Linfoot's *Picture Making in Pen and Ink*, Miller's *Essentials of Perspective* and Delamotte's *Art of Sketching from Nature;* and with an experienced architect or artist, if possible, but otherwise by himself, master the principles and act on the instructions of these writers.

22. Since the camera makes it, fortunately, no longer essential that a civil engineer should be a landscape artist as well, his free-hand work has become more restricted in its scope and more rigid in its character, and like that of the machine designer it may properly be called *technical*, from its object. Yet to attain a sufficient degree of skill in it for all practical and commercial purposes is possible to all, and among them many who could never hope to produce artistic results. It is confined mainly to the making of *working sketches*, *conventional representations* and *free-hand lettering*, and the equipment therefor consists of a pencil of medium grade as to hardness; lettering pens—Falcon or Gillott's 303, with Miller Bros. "Carbon" pen No. 4; either a note-book or a sketch-block or pad; also the following for sketching from measurement: a two-foot pocket-rule; calipers, both external and internal, for taking outside and inside diameters; a pair of pencil compasses for making an occasional circle too large to be drawn absolutely free-hand; and a steel tape-measure for large work, if one can have assistance in taking notes, but otherwise a long rod graduated to eighths.

23. In the evolution of a machine or other engineering project the designer places his ideas on paper in the form of rough and mainly free-hand sketches, beginning with a general outline, or "skeleton" drawing of the whole, on as large a scale as possible, then filling in the details, separate —and larger—drawings of which are later made to exact scale. While such preliminary sketches are not drawn literally "to scale" it is obviously desirable that something like the relative proportions should be preserved and that the closer the approximation thereto the clearer the idea they will give to the draughtsman or workman who has to work from them. A habit of close observation must therefore be cultivated, of analysis of form and of relative direction and proportion, by all who would succeed in draughting, whether as designers or merely as copyists of existing constructions. While the beginner belongs necessarily in the latter category he must not forget that his aim should be to place himself in the ranks of the former, both by a thorough mastery of the fundamental theory that lies back of all correct design and by such training of the hand as shall facilitate the graphic expression of his ideas. To that end he should improve every opportunity to put in practice the following instructions as to

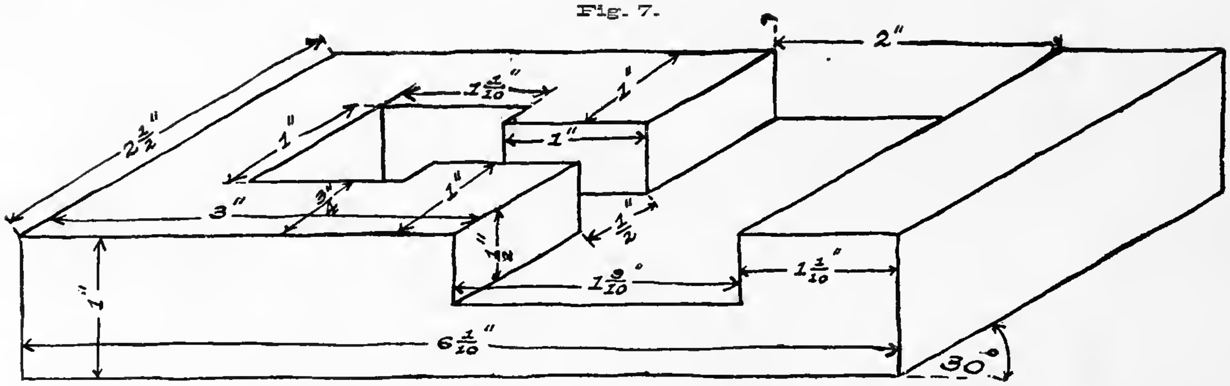
SKETCHING FROM MEASUREMENT,

as each structure sketched and measured will not only give exercise to the hand but also prove a valuable object lesson in the proportioning of parts and the modes of their assemblage.

A free-hand sketch may be as good a working drawing as the exactly scaled—and usually *inked*—drawing that is generally made from it to be sent to the shop.

While several views are usually required, yet for objects of not too complicated form, and whose lines lie mainly in mutually perpendicular directions, the method of representation illustrated by Fig. 7, is admirably adapted,* and obviates all necessity for additional sketches. It is an *oblique projection*

Fig. 7.




FREE-HAND SKETCH OF TIMBER FRAMING.

(Art. 17) the theory of whose construction will be found in a subsequent chapter, but with regard to which it is sufficient at this point to say that the right angles of the front face are seen in their true form, while the other right angles are shown either of 30°, 60°, or 120°; although almost any oblique angle will give the same general effect and may be adopted. Lines parallel to each other on the object are also parallel in the drawing.

Draw first the front face, whose angles are seen in their true form; then run the oblique lines off in the direction which will give the best view. (Refer to Figs. 42, 44, 45 and 46.)

24. While Fig. 7 gives almost the pictorial effect of a true perspective and the object requires no other description, yet for complicated and irregular forms it gives place to the plan-and-elevation mode of representation, the plan being a *top* and the elevation a *front* view of the object. And

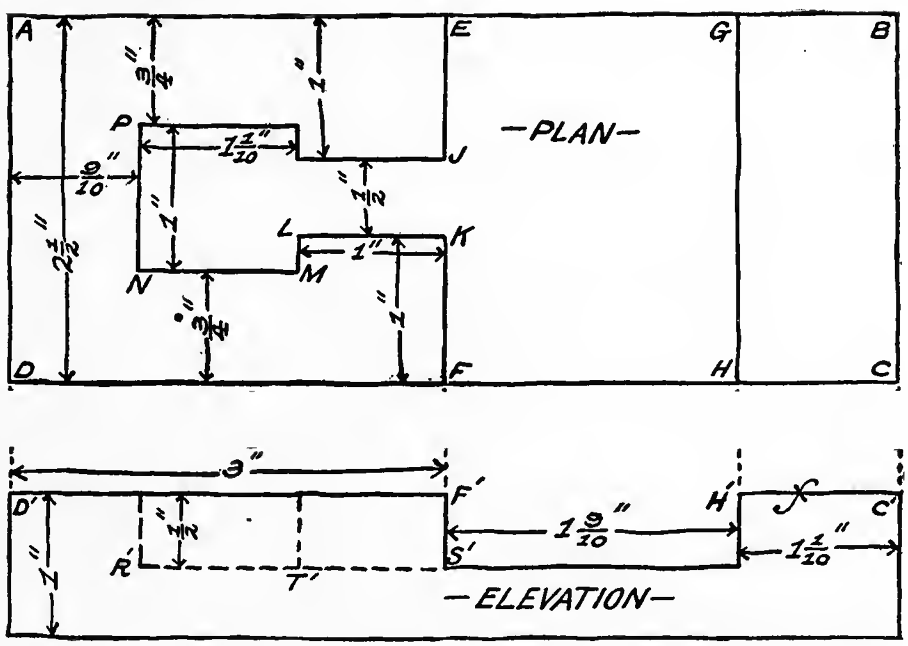
* The figures in this chapter are photo-reproductions of free-hand work and are intended not only to illustrate the text but also to set a reasonable standard for sketch-notes.

if two views are not enough for clearness as many more should be added as seem necessary, including what are called *sections*, which represent the object as if cut apart by a plane, separated and a view obtained perpendicular to the cutting plane, showing the internal arrangement and shape of parts.

In Fig. 8 we have the same object as in Fig. 7, but represented by the method just mentioned. The front view (elevation) is evidently the same in both Figs. 7 and 8, except that in the latter we indicate by dotted lines the hidden recess which is in full sight in Fig. 7.

The view of the top is placed *at the top* in conformity to the now quite general practice as to location, viz., grouping the various sketches about the elevation, so that the view of the left end is *at the left*, of the right *at the right*, etc.

Fig. 8.




FREE-HAND SKETCH OF TIMBER FRAMING, IN PLAN AND ELEVATION.

In these views, which fall under Art. 19 as to theoretical construction, entire surfaces are *projected* as straight lines, as *G B C H* in the straight line *H′ C′*. Were this a metallic surface and "*finished*" or "*machined*" to smoothness, as distinguished from the surface of a rough casting, that fact would be denoted by an "*f*" on the line *H′ C′* which represents the entire surface, the cross-line of the "*f*" cutting the line obliquely, as shown.

CENTRE-LINES.—DIMENSIONING.

25. *Dimensioning.* In sketching, centre-lines and all important centres should be located first, and measurements taken from them or from finished surfaces.

Feet and inches are abbreviated to "Ft.," and "In.," as 4 Ft. $6\frac{3}{8}$ In.: also written 4′ $6\frac{3}{8}''$, and occasionally 4 Ft. $6\frac{3}{8}''$. A dimension should not be written as an improper fraction, $\frac{13}{8}''$ for example, but as a mixed number, $1\frac{5}{8}''$. Fractions should have *horizontal* dividing lines.

Not only should dimensions of successive parts be given but an "over-all" dimension, which, it need hardly be said, should sustain the axiom regarding the whole and the sum of its parts.

Dimensions should read in line with the line they are on, and either from the bottom or the right hand.

The arrow tips should touch the lines between which a distance is given.

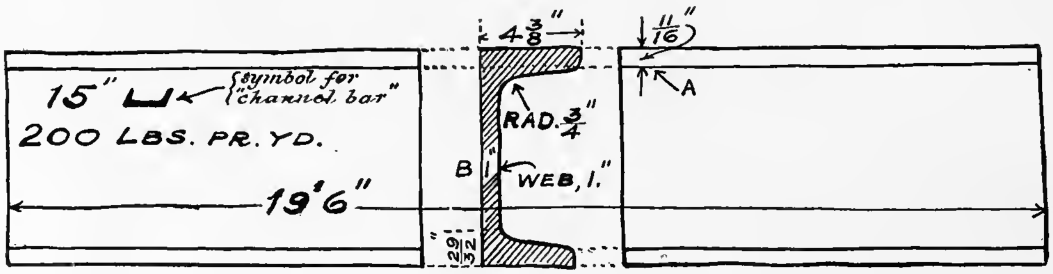
Extension lines should be drawn and the dimension given *outside* the drawing whenever such course will add to the clearness. (See $D' F'$, Fig. 8.)

An opening should always be left in the dimension line for the figures.

In case of very small dimensions the arrow tips may be located outside the lines, as in Fig. 9, and the dimension indicated by an arrow, as at $A$, or inserted as at $B$ if there is room.

Should a piece of *uniform cross-section* (as, for example, a rail, angle-iron, channel bar, Phœnix column or other form of structural iron) be too long to be represented in its proper relative length on the sketch it may be broken as in Fig. 9, and the *form* of the section (which in the case supposed will be the same as an *end view*) may be inserted with its dimensions, as in the shaded figure. If the kind of bar and the number of pounds per yard are known the dimensions can be obtained by reference to the handbook issued by the manufacturers.

Fig. 9.




FREE-HAND SKETCH OF A CHANNEL BAR.

The same dimension should not appear on each view, but each dimension must be given at least once on some view.

*Notes on Riveted Work, Pins, Bolts, Screws and Nuts.* In riveted work the "pitch" of the rivets, i. e., their distance from centre to centre ("c. to c.") should be noted, as also that between centre lines or rows, and of the latter from main centre lines. Similarly for bolts and holes. If the latter are located in a circle note the diameter of the circle containing their centres. Note that a hole for a rivet is usually about one-half the diameter of the forged head.

In measuring nuts take the width between parallel sides ("width across the flats") and abbreviate for the shape, as "sq.," "hex.," "oct."

For a piece of cylindrical shape a frequently used symbol is the circle, as 4″ ○ (read "four inches, round," *not around*,) for 4″ diameter; but it is even clearer to use the abbreviation of the latter word, viz., "diam."

In taking notes on bolts and screws the outside diameter is sufficient if they are "standard," that is, proportioned after either the Sellers (U. S. Standard) or Whitworth (English Standard) systems, as the proportions of heads and nuts, number of threads to the inch, etc., can be obtained from the tables in the Appendix. If not "standard" note the number of threads to the inch. Record whether a screw is right- or left-handed. If right-handed it will advance if turned clock-wise. The shape of thread, whether triangular or square, would also be noted.

*Notes on Gearing.* On cog, or "gear," wheels obtain the distance between centres and the number of teeth on each wheel. The remaining data are then obtained by calculation.

*Bridge Notes.* In taking bridge notes there would be required general sketches of front and end view; of the flooring system, showing arrangement of tracks, ties, guard-beam and side-walk; a cross-section; also detail drawings of the top and foot of each post-connection in one longitudinal line from one end to the middle of the structure. In case of a double-track bridge the outside rows of posts are alike but differ from those of the middle truss.

sections they may employ *burnt umber* undertone for the earthy bed, *pale blue* or *india ink* tint for the rock, and *prussian blue* for the water lines.

### FREE-HAND LETTERING.

27. Although later on in this work an entire chapter is devoted to the subject of lettering, yet at this point a word should be said regarding those forms of letters which ought to be mastered, early in a draughting course, as the most serviceable to the practical worker.

Fig. 12.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

*A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &*
*1 2 3 4 5 6 7 8 9 0*

The first, known as the *Gothic*, is the simplest form of letter, and is illustrated in both its *vertical* and *inclined* (or *Italic*) forms in Fig. 12. It is much used in dimensioning, as well as for titles. The lettering and numerals are Gothic in Figs. 7 and 8, with the exception of the 1 and 4, which, by the addition of feet, are no longer a pure form although enhanced in appearance.

Fig 12 (a).

*935 6/7*

In Fig. 12 (a) some modifications of the forms of certain numerals are shown; also the omission of the dividing line in a mixed number, as is customary in some offices.

For Gothic letters and for all others in which there is to be no shading it is well to use a pen with a blunt end, preferably "ball-pointed," but otherwise a medium stub, like Miller Bros'. "Carbon" No. 4, which gives the desired result when used on a smooth surface and without undue pressure.

Fig. 13 illustrates the Italic (or *inclined*) form of a letter which when vertical is known as the *Roman*. The *Roman* and *Italic Roman* are much used on Government and other map work, and in

Fig. 13.

*A B C D E F G H I J K L M N O P Q R S*

*1 2 3 4 5 T U V W X Y Z 6 7 8 9 0*

*a b c d e f g h i j k l m n o p q r s t u v w x y z*

the draughting offices of many prominent mechanical engineers. Regarding them the student may profitably read Arts. 260–262. Make the spaces between letters as nearly uniform as possible, and the small letters usually about three-fifths the height of the capitals in the same line.

For Roman and other forms of letter requiring shading use a fine pen; Gillott's No. 303 for small work, and a "Falcon" pen for larger.

A form of letter much used in Europe and growing in favor here is the *Soennecken Round Writing*, referred to more particularly in Art. 265 and illustrated by a complete alphabet in the Appendix. The text-book and special pens required for it can be ordered through any dealer in draughtsmen's supplies.

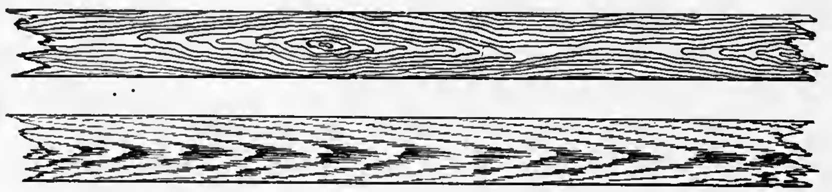
All notes should be taken on as large a scale as possible, and so indexed that drawings of parts may readily be understood in their relation to the whole.

The foregoing hints might be considerably extended to embrace other and special cases, but experience will prove a sufficient teacher if the student will act on the suggestions given, and will remember that to get an excess of data is to err on the side of safety. It need hardly be added that what has preceded is intended to be merely a partial summary of the instructions which would be given in the more or less brief practice in technical sketching which, presumably, constitutes a part of every course in Graphics; and that unless the draughtsman can be under the direction of a teacher he will be able to sketch much more intelligently after studying more of the theory involved in Mechanical Drawing and given in the later pages of this work.

### CONVENTIONAL REPRESENTATIONS.

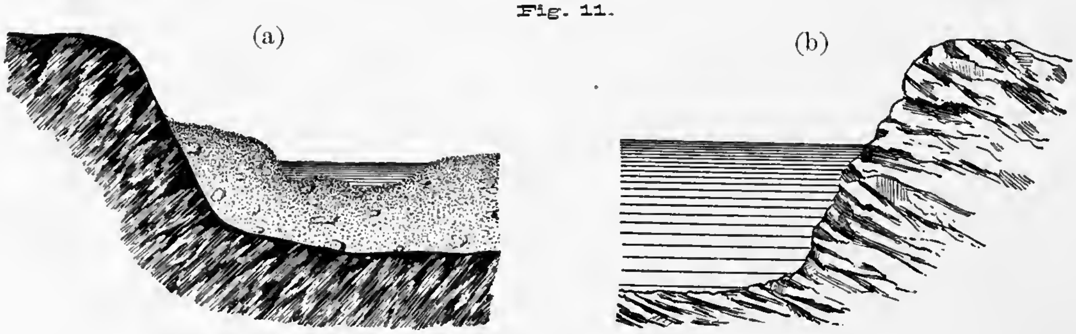
26. Conventional representations of the natural features of the country or of the materials of construction are so called on the assumption, none too well founded, that the engineering profession

Fig. 10.



has agreed in convention that they shall indicate that which they also more or less resemble. While there is no universal agreement in this matter there is usually but little ambiguity in their use, especially in those that are drawn free-hand, since in them there can be a nearer approach to the natural appearance. This is well illustrated by Figs. 10 and 11.

Fig. 11.




In addition to a rock section Fig. 11 (a) shows the method of indicating a mud or sand bed with small random boulders.

Water either in section or as a receding surface may be shown by parallel lines, the spaces between them increasing gradually.

Conventional representations of wood, masonry and the metals will be found in Chapter VI, after hints on coloring have been given, the foregoing figures appearing at this point merely to illustrate, in black and white, one of the important divisions of technical free-hand work. Those, however, who have already had some practice in drawing may undertake them either with pen and ink or in colors, in the latter case observing the instructions of Arts. 237–241 for wood, while for the river

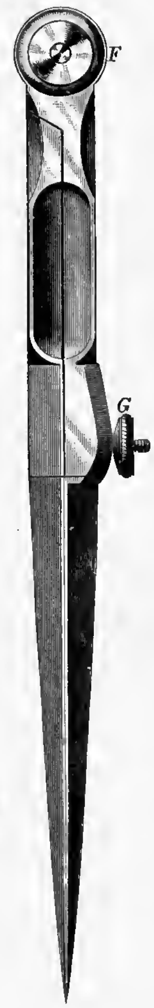
# CHAPTER III.

## DRAWING INSTRUMENTS AND MATERIALS.—INSTRUCTIONS AS TO USE.—GENERAL PRELIMINARIES AND TECHNICALITIES.

28. The draughtsman's equipment for graphical work should be the best consistent with his means. It is mistaken economy to buy inferior instruments. The best obtainable will be found in the end to have been the cheapest.

Fig. 14.


Fig. 15.




The set of instruments illustrated in the following figures contains only those which may be considered absolutely essential for the beginner.

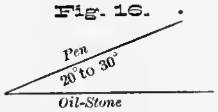
### THE DRAWING PEN.

The right line pen (Fig. 14) is ordinarily used for drawing straight lines, with either a rule or triangle to guide it; but it is also employed for the drawing of curves when directed in its motion by curves of wood or hard rubber. For average work a pen about five inches long is best.

The figure illustrates the most approved type, i. e., made from a single piece of steel. The distance between its points, or "nibs," is adjustable by means of the screw *H*. An older form of pen has the outer blade connected with the inner by a hinge. The convenience with which such a pen may be cleaned is more than offset by the certainty that it will not do satisfactory work after the joint has become in the slightest degree loose and inaccurate through wear.

29. If the points wear unequally or become blunt the draughtsman may sharpen them readily himself upon a fine oil-stone. The process is as follows: Screw up the blades till they nearly touch. Incline the pen at a small angle to the surface of the stone and draw it lightly from left to right (supposing the initial position as in Fig. 16). Before reaching the right end of the stone begin turning the pen in a plane perpendicular to the surface, and draw in the opposite direction at the same angle. After frequent examination and trial, to see that the blades have become equal in length and similarly rounded, the process is completed by lightly dressing the outside of each blade separately upon the stone. No grinding should be done on the inside of the blade. Any "burr" or rough edge resulting from the operation may be removed with fine emery paper. For the best results, obtained in the shortest possible time, a magnifying glass should be used. The student should take particular notice of the shape of the pen when new, as a standard to be aimed at when compelled to act on the above suggestions.

Fig. 16.

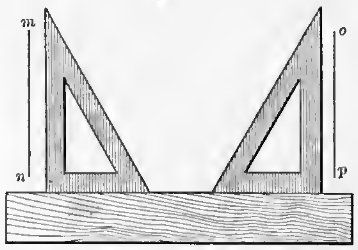



30. The pen may be supplied with ink by means of an ordinary writing pen dipped in the ink and then passed between the blades; or by using in the same manner a strip of Bristol board about a quarter of an inch in width. Should any fresh ink get on the outside of the pen it must

be removed; otherwise it will be transferred to the edge of the rule and thence to the paper, causing a blot.

31. As with the pencil, so with the pen, horizontal lines are to be drawn *from left to right*, while vertical or inclined lines are drawn either from or toward the worker, according to the position of the guiding edge with respect to the line to be drawn. If the line were $m\,n$, Fig. 17, the motion would be away from the draughtsman, i. e., from $n$ toward $m$; while $o\,p$ would be drawn *toward* the worker, being on the right of the triangle.

Fig. 17.




32. To make a sharply defined, clean-cut line—the only kind allowable—the pen should be held lightly but firmly with one blade resting against the guiding edge, and with both points resting equally upon the paper so that they may wear at the same rate.

33. The inclination of the pen to the paper may best be about 70°. When properly held the pen will make a line about a fortieth of an inch from the edge of the rule or triangle, leaving visible a white line of the paper of that width. If, then, we wish to connect two points by an inked straight line, the rule must be so placed that its edge will be from them the distance indicated.

It need hardly be said that a *drawing*-pen should not be *pushed.*

The more frequently the draughtsman will take the trouble to clean out the point of the pen and supply fresh ink the more satisfactory results will he obtain. When through with the pen clean it carefully, and lay it away with the points not in contact. Equal care should be taken of all the instruments, and for cleaning them nothing is superior to chamois skin.

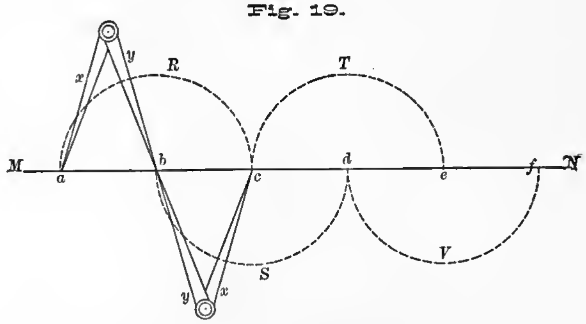
DIVIDERS.

34. The hair-spring dividers (Fig. 15) are employed in dividing lines and spacing off distances, and are capable of the most delicate adjustment by means of the screw $G$ and spring in one of the legs. When but one pair of dividers is purchased the kind illustrated should have the preference over plain dividers, which lack the spring. It will, however, be frequently found convenient to have at hand a pair of each. Should the joint at $F$ become loose through wear it can be tightened by means of a key having two projections which fit into the holes shown in the joint.

35. In spacing off distances the pressure exerted should be the slightest consistent with the location of a point, the puncture to be merely in the surface of the paper and the points determined by lightly pencilled circles about them, thus  . In laying off several equal distances along a line all the arcs described by one leg of the dividers should be on the same side of the line. Thus, in Fig. 19, with $b$ the first centre of turning, the leg $x$ describes the

Fig. 18.

Fig. 19.

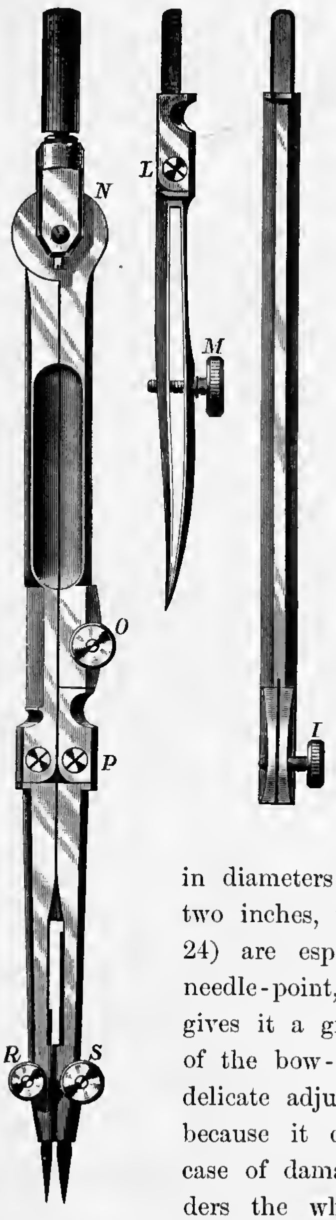



arc $R$, then rests and pivots on $c$ while the leg $y$ describes the arc $S$; $x$ then traces arc $T$, etc.

36. The compasses (Fig. 20) resemble the dividers in form and *may* be used to perform the same office, but are usually employed for the drawing of circles. Unlike the dividers one or both of the legs of compasses are detachable. Those illustrated have one permanent leg, with pivot or "needle-point" adjustable by means of screw *R*. The other leg is detachable by turning the screw *O*, when the pen leg *L M* (Fig. 21) may be inserted for ink work; or, where large work is involved, the lengthening bar on the right (Fig. 22) may be first attached at *O* and the pencil or pen leg then inserted at *I*. The metallic point held by screw *S* is usually replaced by a hard lead, sharpened as indicated in Art 54.

COMPASS SET.

Fig. 20. Fig. 21. Fig. 22.




37. When in use the legs should be bent at the joints *P* and *L*, so that they will be perpendicular to the paper when the compasses are held in a vertical plane. The turning may be in either direction, but is usually "clock-wise;" and the compasses may be slightly inclined toward the direction of turning. When so used, and if no undue pressure be exerted on the pivot leg, there should be but the slightest puncture at the centre, while the pen points having rested equally upon the paper have sustained equal wear, and the resulting line has been sharply defined on both sides. Obviously the legs must be re-adjusted as to angle, for any material change in the size of the circles wanted.

The compasses should be held and turned by the milled head which projects above the joint *N*.

Dividers and compasses should open and shut with an absolutely uniform motion and somewhat stiffly.

BOW-PENCIL AND PEN.

Fig. 23.

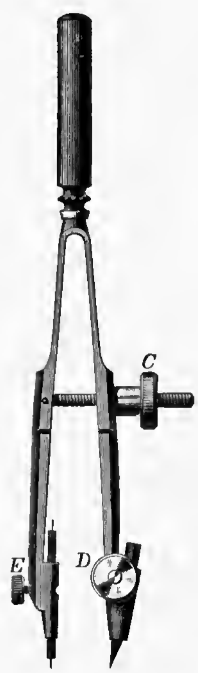
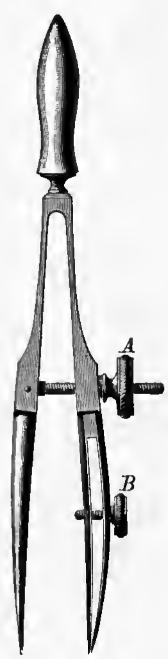



Fig. 24.




38. For extremely accurate work, in diameters from one-sixteenth of an inch to about two inches, the *bow-pencil* (Fig. 23) and *bow-pen* (Fig. 24) are especially adapted. The pencil-bow has a needle-point, adjustable by means of screw *E*, which gives it a great advantage over the fixed pivot-point of the bow-pen, not alone in that it permits of more delicate adjustment for unusually small work but also because it can be easily replaced by a new one in case of damage; whereas an injury to the other renders the whole instrument useless. For very small circles the needle-point should project *very slightly* beyond the pen-point; theoretically by only the extremely small distance the needle-point is expected to sink into the paper.

The spring of either bow should be strong; otherwise an attempt at a circle will result in a spiral.

It will save wear upon the threads of the milled heads *A* and *C* if the draughtsman will press the legs of the bow together with his left hand and run the head up loosely on the screw with his right.

39. To the above described—which we may call the *minimum* set of instruments—might be advantageously added a pair of bow-spacers (small dividers shaped like Fig. 24); beam-compasses, for extra large circles; parallel-rule; proportional dividers, and an extra—and larger—right-line pen.

40. The remainder of the *necessary* equipment consists of paper; a drawing-board; T-rule; triangles or "set squares;" scales; pencils; India ink; water colors; saucers for mixing ink or colors; brushes; water-glass and sponge; irregular (or "French") curves; india rubber; erasing knife; protractor; file for sharpening pencils, or a pad of fine emery or sand paper; thumb-tacks (or "drawing-pins"); horn centre, for making a large number of concentric circles.

PAPER AND TRACING CLOTH.

41. Drawing paper may be purchased by the sheet or roll and either unmounted or mounted, i. e., "backed" by muslin or heavy card-board. Smooth or "hot-pressed" paper is best for drawings in line-work only; but the rougher surfaced, or "cold-pressed," should always be employed when brush-work in ink or colors is involved: in the latter case, also, either mounted paper should be used or the sheets "stretched" by the process described in Art. 44.

42. The names and sizes of sheets are:—

| | |
|---|---|
| Cap 13 × 17 | Elephant 23 × 28 |
| Demi 15 × 20 | Atlas 26 × 34 |
| Medium 17 × 22 | Columbia 23 × 35 |
| Royal 19 × 24 | Double Elephant 27 × 40 |
| Super Royal 19 × 27 | Antiquarian 31 × 53 |

Imperial 22 × 30

43. There are many makes of first-class papers, but the best known and still probably the most used is Whatman's. The draughtsman's choice of paper must, however, be determined largely by the value of the drawing to be made upon it, and by the probable usage to which it will be subjected.

Where several copies of one drawing were desired it has been a general practice to make the original, or "construction" drawing, with the *pencil*, on paper of medium grade, then to lay over it a sheet of *tracing-cloth* and copy upon it, in ink, the lines underneath. Upon placing the tracing cloth over a sheet of sensitized paper, exposing both to the light and then immersing the sensitive paper in water, a copy or print of the drawing was found upon the sheet, in white lines on a blue ground—the well-known *blue-print*. The time of the draughtsman may, however, be economized, as also his purse, by making the original drawing in ink upon Crane's *Bond* paper, which combines in a remarkable degree the qualities of transparency and toughness. About as clear blue-prints can be made with it as with tracing-cloth, yet it will stand severe usage in the shop or the drafting-room.

Better papers may yet be manufactured for such purposes, and the progressive draughtsman will be on the alert to avail himself of these as of all genuine improvements upon the materials and instruments before employed.

44. To stretch paper tightly upon the board lay the sheet right side up,* place the long rule with its edge about one-half inch back from each edge of the paper in turn, and fold up against it a margin of that width. Then *thoroughly* dampen the *back* of the paper with a full sponge, except on the folded margins. Turning the paper again face up gum the margins with strong mucilage or glue, and quickly but firmly press *opposite* edges down simultaneously, long sides first, exerting at the same time a slight outward pressure with the hands to bring the paper down somewhat closer to

*The "right side" of a sheet is, presumably, that toward one when—on holding it up to the light—the manufacturer's name, in water-mark, reads correctly.

the board. Until the gum "sets," so that the paper adheres perfectly where it should, the latter should not shrink; hence the necessity for so completely soaking it at first. The sponge may be applied to the *face* of the paper provided it is not *rubbed* over the surface, so as to damage it. The stretch should be horizontal when drying, and no excess of water should be left standing on the surface; otherwise a water-mark will form at the edge of each pool.

45. When tracing-*cloth* is used it must be fastened smoothly, with thumb-tacks, over the drawing to be copied, and the ink lining done upon the glazed side, any brush work that may be required—either in ink or colors—being always done upon the dull side of the cloth after the outlining has been completed.

If the glazed surface be first dusted with powdered pipe-clay applied with chamois skin it will take the ink much more readily.

When erasure is necessary use the rubber, after which the surface may be restored for further pen-work by rubbing it with soapstone.

Tracing-cloth, like drawing paper, is most convenient to work upon if perfectly flat. When either has been purchased by the roll it should therefore be cut in sheets and laid away for some time in drawers to become flat before needed for use.

### DRAWING BOARD.

46. The drawing board should be slightly larger than the paper for which it is designed and of the most thoroughly seasoned material, preferably some *soft* wood, as pine, to facilitate the use of the drawing-pins or thumb-tacks. To prevent warping it should have battens of hard wood dove-tailed into it across the back, transversely to its length. The back of the board should be grooved longitudinally to a depth equal to half the thickness of the wood, which weakens the board transversely and to that degree facilitates the stiffening action of the battens.

For work of moderate size, on stretched paper, yet without the use of mucilage, the "panel" board is recommended, provided that both frame and panel are made of the best seasoned hard wood.

It will be found convenient for each student in a technical school to possess two boards, one 20″ × 28″ for paper of Super Royal size, which is suitable for much of a beginner's work, and another 28″ × 41″ for Double Elephant sheets (about twice Super Royal size) which are well adapted to large drawings of machinery, bridges, etc. A large board may of course be used for small sheets, and the expense of getting a second board avoided; but it is often a great convenience to have a medium-sized board, especially in case the student desires to do some work outside the draughting-room.

### THE T-RULE.

47. The T-rule should be slightly shorter than the drawing board. Its head and blade must have absolutely straight edges, and be so rigidly combined as to admit of no lateral play of the latter in the former. The head should also be so fastened to the blade as to be level with the surface of the board. This permits the triangles to slide freely over the head, a great convenience when the lines of the drawing run close to the edge of the paper. (See Fig. 32.)

The head of the T-rule should always be used along the left-hand edge of the drawing board.

### TRIANGLES.

48. Triangles, or "set-squares" as they are also called, can be obtained in various materials, as hard rubber, celluloid, pear-wood, mahogany and steel; and either solid (Fig. 25) or open (Fig. 26). The open triangles are preferable, and two are required, one with acute angles of 30° and 60°, the other with 45° angles. Hard rubber has an advantage over metal or wood, the latter being likely to warp and the former to rust, unless plated. Celluloid is transparent and the most cleanly of all.

The most frequently recurring problems involving the use of the triangles are the following:—

Fig. 25.

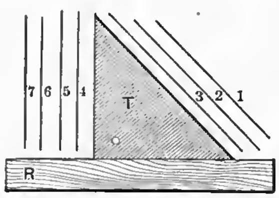
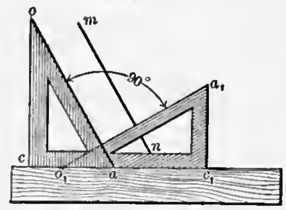


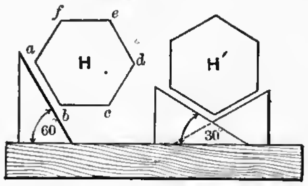

Fig. 26.




49. *To draw parallel lines* place either of the edges against another triangle or the T-rule. If then moved along, in either direction, each of the other edges will take a series of parallel positions.

50. *To draw a line perpendicular to a given line* place the *hypothenuse* of the triangle, $o\ a$, (Fig. 26), so as to coincide with or be parallel to the given line; then a rule or another triangle against the base. By then turning the triangle so that the other side, $o\ c$, of its right angle shall be against the rule, as at $o_1\ c_1$, the hypothenuse will be found perpendicular to its first position and therefore to the given line.

Fig. 27.




51. *To construct regular hexagons* place the *shortest* side of the 60° triangle against the rule (Fig. 27) if two sides are to be *horizontal*, as $f\ e$ and $b\ c$ of hexagon *H*. For vertical sides, as in H′, the position of the triangle is evident. By making $a\ b$ indefinite at first, and knowing $b\ c$—the length of a side, we may obtain $a$ by an arc, centre $b$, radius $b\ c$.

If the inscribed circles were given, the hexagons might also be obtained by drawing a series of tangents to the circles, with the rule and triangles in the positions indicated.

THE SCALE.

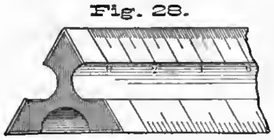
52. But rarely can a drawing be made of the same size as the object, or "full-size," as it is called; the lines of the drawing, therefore, usually bear a certain ratio to those of the object. This ratio is called the *scale* and should invariably be indicated.

If six inches on the drawing represent one foot on the object the scale is *one-half* and might be variously indicated, thus: SCALE $\frac{1}{2}$; SCALE 1:2; *SCALE 6 IN. = 1 FT. SCALE 6″ = 1′.*

At one foot to the inch any line of the drawing would be one-twelfth the actual size, and the fact indicated in either of the ways just illustrated.

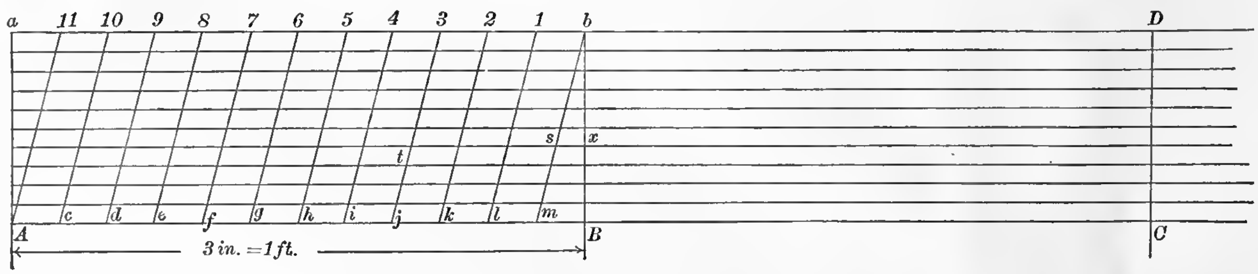
Although it is a simple matter for the draughtsman to make a scale for himself for any particular case yet scales can be purchased in great variety, the most serviceable of which for the usual range of work is of box-wood, 12″ long, (or 18″, if for large work) of the form illustrated by Fig. 28, and graduated $\frac{3}{32} : \frac{3}{16} : \frac{1}{8} : \frac{1}{4} : \frac{3}{8} : \frac{1}{2} : \frac{3}{4} : 1 : 1\frac{1}{2} : 3$ inches to the foot. This is known as the *architect's* scale in contradistinction to the *engineer's*, which is decimally graduated. It will, however, be frequently convenient to have at hand the latter as well as the former.

Fig. 28.



When in use it should be laid along the line to be spaced, and a light dot made upon the

Fig. 29.




paper with the pencil, opposite the proper division on the graduated edge. A distance should rarely

be transferred from the scale to the drawing by the dividers, as such procedure damages the scale if not the paper.

53. For special cases *diagonal* scales can readily be constructed. If, for example, a scale of 3 inches to the foot is needed and measuring to *fortieths of inches*, draw eleven equidistant, parallel lines, enclosing ten equal spaces, as in Fig. 29, and from the end $A$ lay off $A\,B$, $B\,C$, etc., each 3 inches and representing a foot. Then twelve parallel diagonal lines in the first space intercept quarter-inch spaces on $A\,B$ or $a\,b$, each representative of an inch. There being ten equal spaces between $B$ and $b$, the distance $s\,x$, of the diagonal $b\,m$ from the vertical $b\,B$, taken on any horizontal line $s\,x$, is as many tenths of the space $m\,B$ as there are spaces between $s\,x$ and $b$; *six*, in this case. The principle of construction may be generalized as follows:—

The distance apart of the vertical lines represents the units of the scale, whether inches, feet, rods or miles. Except for decimal graduation divide the left-hand space at top and bottom into as many spaces as there are units of the next lower denomination in one of the original units (feet, for yards as units; inches in case of feet, etc.). Join the points of division by diagonal lines; and, if $\frac{1}{x}$ is the smallest fraction that the scale is designed to give, rule $x + 1$ equidistant horizontal lines, giving $x$ equal horizontal spaces. The scale will then read to $\frac{1}{x}$th of the intermediate denomination of the scale.

When a scale is properly used, the spaces on it which represent feet and inches are treated as if they were such in fact. On a scale of one-eighth actual size the edge graduated $1\frac{1}{2}$ inches to the foot would be employed; each $1\frac{1}{2}$ inch space on the scale would be read as if it were a foot; and ten inches, for example, would be ten of the eighth-inch spaces, each of which is to represent an inch of the original line being scaled. The usual error of beginners would be to divide each original dimension by eight and lay off the result, actual size. The former method is the more expeditious.

### THE PENCILS.

54. For construction lines afterward to be inked the pencils should be of *hard* lead, grade 6H if Fabers or VVH if Dixon's. The pencilling should be *light.* It is easy to make a groove in the paper by exerting too great pressure when using a hard lead. The hexagonal form of pencil is usually indicative of the finest quality, and has an advantage over the cylindrical in not rolling off when on a board that is slightly inclined.

Somewhat softer pencils should be used for drawings afterward to be traced, and for the preliminary free-hand sketches from which exact drawings are to be made; also in free-hand lettering.

Sharpen to a *chisel* edge for work along the edges of the T-rule or triangles, but use another pencil with *coned* point for marking off distances with a scale, locating centres and other isolated points, and for free-hand lettering; also sharpen the compass leads to a point. Use the knife for cutting the wood of the pencil, beginning at least an inch from the end. Leave the lead exposed for a quarter of an inch and shape it as desired, either with a knive or on a fine file, or a pad of emery paper.

### THE INK.

55. Although for many purposes some of the liquid drawing-inks now in the market, particularly Higgins', answer admirably, yet for the best results, either with pen or brush, the draughtsman should mix the ink himself with a stick of India—or, more correctly, *China* ink, selecting one of the higher-priced cakes, of rectangular cross-section. The best will show a lustrous, almost iridescent fracture, and will have a smooth, as contrasted with a gritty *feel* when tested by rubbing the moistened finger on the end of the cake.

Sets of saucers, called "nests," designed for the mixing of ink and colors, form an essential part of an equipment. There are usually six in a set and so made that each answers as a cover for the one below it. Placing from fifteen to twenty drops of water in one of these the stick of ink should be rubbed on the saucer with *moderate* pressure.

To properly mix ink requires great patience, as with too great pressure a mixture results having flakes and sand-like particles of ink in it, whereas an absolutely smooth and rather thick, slow-flowing liquid is wanted, whose surface will reflect the face like a mirror. The final test as to sufficiency of grinding is to draw a broad line and let it dry. It should then be a rich jet black, with a slight lustre. The end of the cake must be carefully dried on removing it from the saucer to prevent its flaking, which it will otherwise invariably do.

One may say, almost without qualification, and particularly when for use on tracing-cloth, the thicker the ink the better; but if it should require thinning, on saving it from one day to another—which is possible with the close-fitting saucers described—add a few drops of water, or of ox-gall if for use on a glazed surface.

When the ink has once dried on the saucer no attempt should be made to work it up again into solution. Clean the saucer and start anew.

### WATER COLORS.

56. The ordinary colored writing inks should never be used by the draughtsman. They lack the requisite "body" and are corrosive to the pen. Very good colored drawing inks are now manufactured for line work, but Winsor and Newton's water colors, in the form called "moist," and in "half-pans" are the best if not the most convenient, for color work either with pen or brush. Those most frequently employed in engineering and architectural drawing are Prussian Blue, Carmine, Light Red, Burnt Sienna, Burnt Umber, Vermilion, Gamboge, Yellow Ochre, Chrome Yellow, Payne's Gray and Sepia. For some of their special uses see Art. 73.

Although hardly properly called a color Chinese White may be mentioned at this point as a requisite, and obtainable of the same form and make as the colors above.

### DRAWING-PINS.

57. Drawing-pins or thumb-tacks, for fastening paper upon the board, are of various grades, the best, and at present the cheapest, being made from a single disc of metal one-half inch in diameter, from which a section is partially cut, then bent at right angles to the surface, forming the point of the pin.

### IRREGULAR CURVES.

58. Irregular or French curves, also called *sweeps*, for drawing non-circular arcs, are of great variety, and the draughtsman can hardly have too many of them. They may be either of pear wood or hard rubber. A thoroughly equipped draughting office will have a large stock of these curves, which may be obtained in sets, and are known as railroad curves, ship curves, spirals, ellipses, hyperbolas, parabolas and combination curves. Some very serviceable *flexible* curves are also in the market.

Fig. 30.

If but two are obtained (which would be a minimum stock for a beginner) the forms shown in Fig. 30 will probably prove as serviceable as any. When employing them for inked work the pen should be so turned, as it advances, that its blades will maintain the same relation (parallelism) to the edge of the guiding curve as they ordinarily do to the edge of

the rule. And the student must content himself with drawing slightly less of the curve than might apparently be made with one setting of the sweep, such course being safer in order to avoid too close an approximation to angles in what should be a smooth curve. For the same reason, when placed in a new position, a portion of the irregular curve must coincide with a part of that last inked.

The pencilled curve is usually drawn free-hand, after a number of the points through which it should pass have been definitely located. In sketching a curve free-hand it is much more naturally and smoothly done if the hand is always kept on the concave side of the curve.

### INDIA RUBBER.

59. For erasing pencil-lines and cleaning the paper india rubber is required, that known as "velvet" being recommended for the former purpose, and either "natural" or "sponge" rubber for the latter. Stale bread crumbs are equally good for cleaning the surface of the paper after the lines have been inked, but will damage pencilling to some extent.

One end of the velvet rubber may well be wedge-shaped in order to erase lines without damaging others near them.

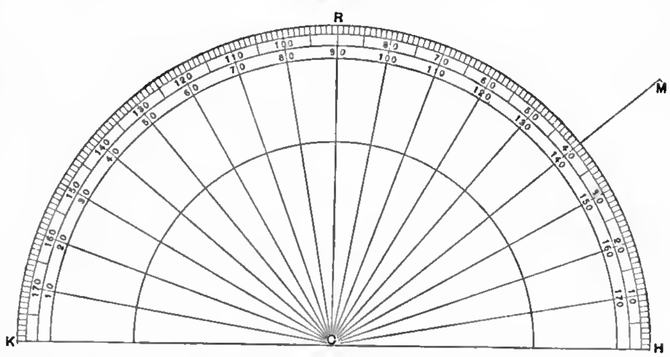
### INK ERASER.

60. The double-edged erasing knife gives the quickest and best results when an inked line is to be removed. The point should rarely be employed. The use of the knife will damage the paper more or less, to partially obviate which rub the surface with the thumb-nail or an ivory knife handle.

### PROTRACTOR.

61. For laying out angles a graduated arc called a "protractor" is used. Various materials are employed in the manufacture of protractors, as metal, horn, celluloid, Bristol board and tracing paper. The two last are quite accurate enough for ordinary purposes, although where the utmost precision is required, one of German silver should be obtained, with a moveable arm and vernier attachment.

Fig. 31.




The graduation may advantageously be to half degrees for average work.

To lay out an angle (say 40°) with a protractor, the radius *C H* (Fig. 31) should be made to coincide with one side of the desired angle; the centre, *C*, with the desired vertex; and a dot made with the pencil opposite division numbered 40 on the graduated edge. The line *M C*, through this point and *C*, completes the construction.

### BRUSHES.

62. Sable-hair brushes are the best for laying flat or graduated tints, with ink or colors, upon small surfaces; while those of camel's hair, large, with a brush at each end of the handle, are better adapted for tinting large surfaces. Reject any brush that does not come to a perfect point on being moistened. Five or six brushes of different sizes are needed.

### PRELIMINARIES TO PRACTICAL WORK.

63. The first work of a draughtsman, like most of his later productions, consists of *line* as distinguished from *brush* work, and for it the paper may be fastened upon the board with thumb-tacks only.

There is no universal standard as to *size* of sheets for drawings. As a rule each draughting office has its own set of standard sizes, and system of preserving and indexing. The columns of the various engineering papers present frequent notes on these points, and the best system of preserving and recording drawings, tracings and corrections is apparently in process of evolution. For the student the best plan is to have all drawings of the same size bound in neat but permanent form at the end of the course. The title-pages, which presumably have also been drawn, will sufficiently distinguish the different sets.

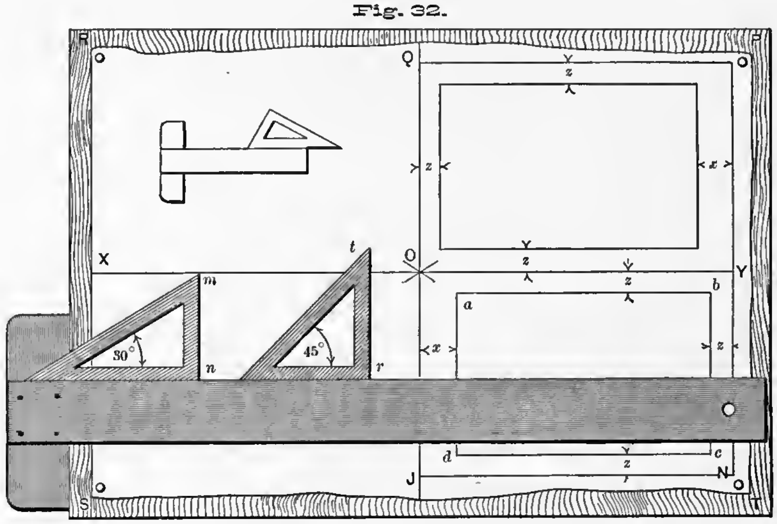
In his elementary work the student may to advantage adopt two sizes of sheets which are considerably employed, 9″ × 13″, and its double, 13″ × 18″; sizes into which a "Super Royal" sheet naturally divides, leaving ample margins for the mucilage in case a "stretch" is to be made.

A "Double Elephant" sheet being twice the size of a "Super Royal" divides equally well into plates of the above size, but is preferable on account of its better quality.

To lay out four rectangles upon the paper locate first the centre (see Fig. 32) by intersecting diagonals, as at *O*. These should *not* be drawn entirely across the sheet, but one of them will necessarily pass a *short* distance each side of the point where the centre lies—judging by the eye alone; the second definitely determines the point. If the T-rule will not reach diagonally from corner to corner of the paper (and it usually will not) the edge may be practically extended by placing a triangle against but *projecting beyond* it, as in the upper left-hand portion of the figure.

The T-rule being placed as shown, with its head *at the left end* of the board—the correct and usual position—draw a horizontal line *X Y*, through the centre just located. The vertical centre line is then to be drawn, with one of the triangles placed as shown in the figure, i. e., so that a side, as *m n* or *t r*, is perpendicular to the edge.

Fig. 32.




It is true that as long as the edges of the board are exactly at right angles with each other we might use the T-rule altogether for drawing mutually perpendicular lines. This condition being, however, rarely realized for any length of time, it has become the custom—a safe one, as long as rule and triangle remain "true"—to use them as stated.

The outer rectangles for the drawings (or "plates," in the language of the technical school) are completed by drawing parallels, as *J N* and *Y N*, to the centre lines, at distances from them of 9″ and 13″ respectively, laid off *from the centre*, *O*.

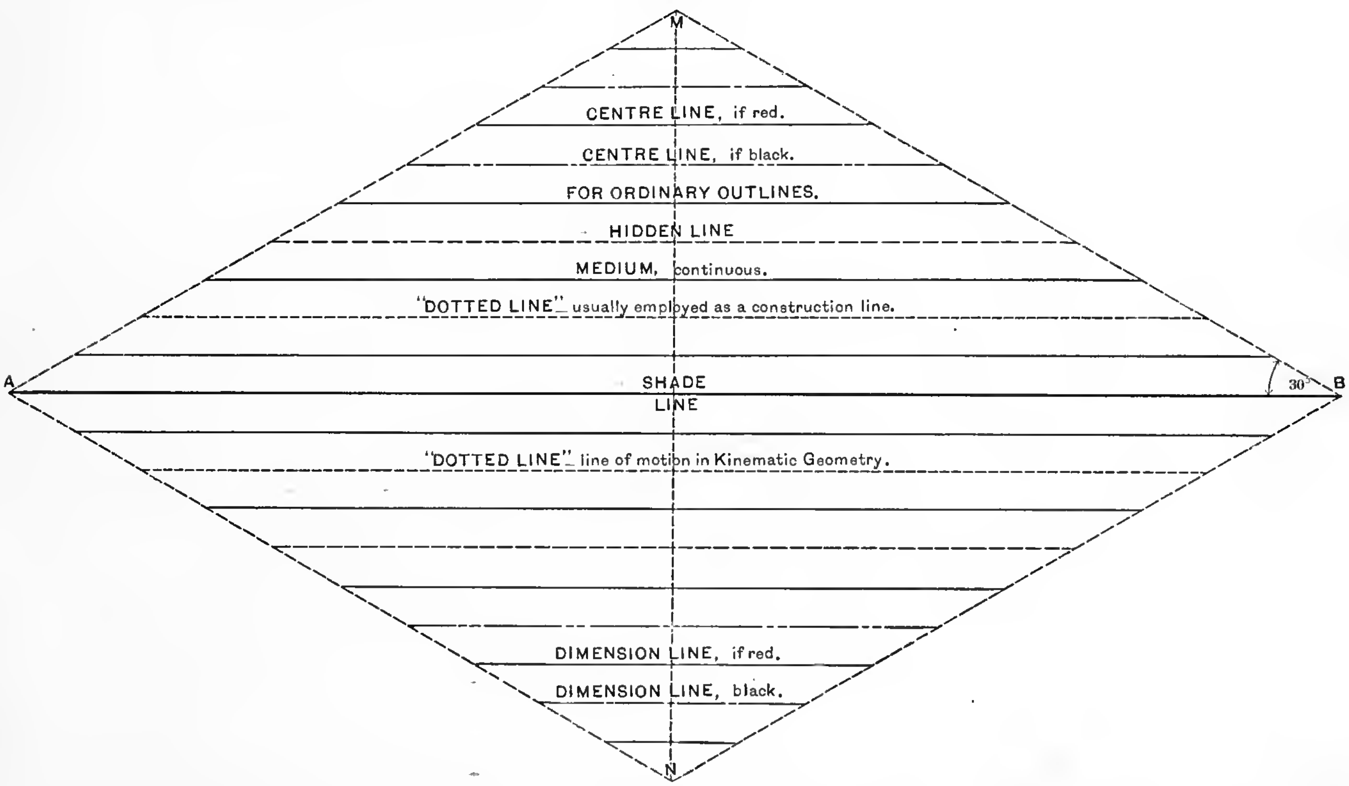
An inner rectangle, as *a b c d*, should be laid out on each plate, with proper margins; usually at least an inch at the top, right and bottom, and an *extra half inch on the left* as an allowance for binding. These margins are indicated by *x* and *z* in the figure, as variables to which any convenient values may be assigned. The broad margin *x* in the upper rectangle will be at the draughtsman's left hand if he turns the board entirely around—as would be natural and convenient—when ready to draw on the rectangle *Q Y*.

## *CHAPTER IV.*

GRADES OF LINES.—LINE TINTING.—LINE SHADING.—CONVENTIONAL SECTION-LINING.—FREQUENTLY RECURRING PLANE PROBLEMS.—MISCELLANEOUS PEN AND COMPASS EXERCISES.

65. Several kinds of lines employed in mechanical drawing are indicated in the figure below. While getting his elementary practice with the ruling-pen the student may group them as shown, or in any other symmetrical arrangement, either original with himself or suggested by other designs.

Fig. 33.




When drawing on tracing cloth or tracing paper, for the purpose of making blue-prints, all the lines will preferably be *black*, and the centre and dimension lines distinguished from others as indicated above, as also by being somewhat finer than those employed for the light outlines of the object. Heavy, opaque, *red* lines may, however, be used, as they will blue-print, though faintly.

There is at present no universal agreement among the members of the engineering profession as to standard dimension and centre lines. Not wishing to add another to the systems already at variance, but preferring to facilitate the securing of the uniformity so desirable, I have presented those for some time employed by the Pennsylvania Railroad and now taught at Cornell University.

The lines of Fig. 33, as also of nearly all the other figures of this work, having been printed from blocks made by the cerographic process (Art. 277), are for the most part too light to serve as examples for machine-shop work. Fig. 80 is a sample of P. R. R. drawing, and is a fair model as to weight of line for working drawings.

A dash-and-three-dot line (not shown in the figure) is considerably used in Descriptive Geometry, either to represent an *auxiliary* plane or an *invisible* trace of *any* plane. (See Fig. 238).

The so-called "dotted" line is actually composed of short dashes. Its use as a "line of motion" was suggested at Cornell.

When colors are used without intent to blue-print they may be drawn as light, continuous lines. Colors will further add to the intelligibility of a drawing if employed for *construction* lines. Even if red dimension lines are used the *arrow heads* should invariably be *black*. They should be drawn *free-hand*, with a writing pen, and their points *touch* the lines between which they give the distance.

66. The utmost accuracy is requisite in pencilling, as the draughtsman should be merely a copyist when using the pen. On a complicated drawing even the *kind* of line should be indicated at the outset, so that no time will be wasted, when inking, in the making of distinctions to which thought has already been given during the process of construction. No unnecessary lines should be drawn, or any exceeding of the intended limit of a line if it can possibly be avoided.

If the work is symmetrical, in whole or in part, draw centre lines first, then main outlines; and continue the work from large parts to small.

The visible lines of an object are to be drawn first; afterward those to be indicated as concealed.

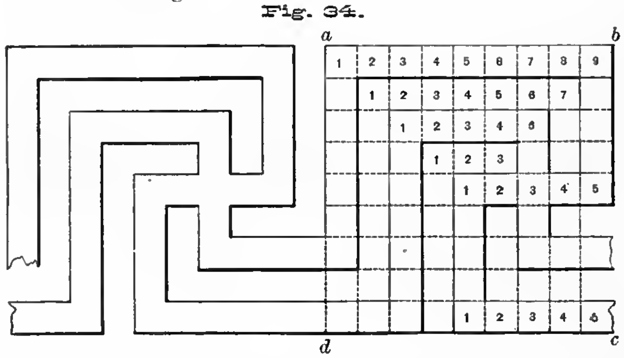
All lines of the same quality may to advantage be drawn with one setting of the pen, to ensure uniformity; and the light outlines before the shade lines.

In drawing arcs and their tangents ink the former first, invariably.

All the inking may best be done at once, although for the sake of clearness, in making a large and complicated drawing, a portion—usually the nearest and visible parts—may be inked, the drawing cleaned, and the pencilling of the construction lines of the remainder continued from that point.

The inking of the centre, dimension and construction lines naturally follows the completion of the main design.

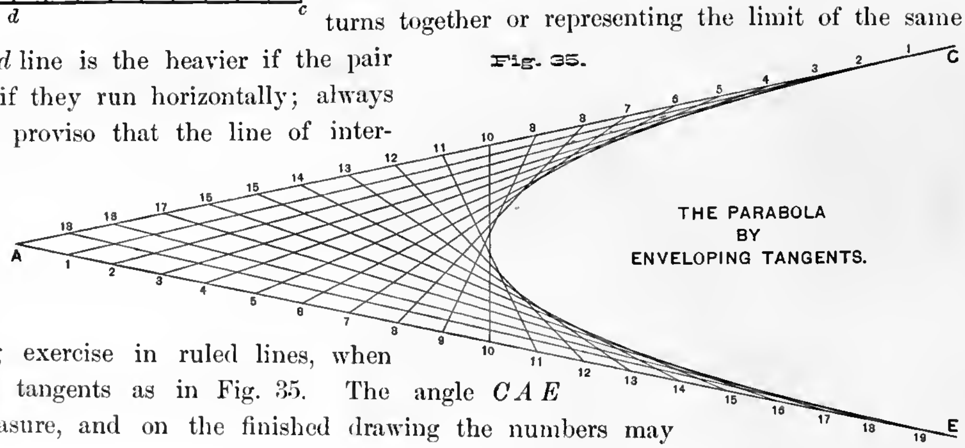
Fig. 34.




67. In Fig. 34 we have a straight-line design usually called the "Greek Fret," and giving the student his first illustration of the use of the "shade line" to bring a drawing out "in relief." The law of the construction will be evident on examination of the numbered squares.

Without entering into the theory of shadows at this point we may state briefly the "shop rule" for drawing shade lines, viz., *right-hand and lower*. That is, of any pair of lines making the same turns together or representing the limit of the same flat surface, the *right-hand* line is the heavier if the pair is vertical, but the *lower* if they run horizontally; always subject, however, to the proviso that the line of intersection of two illuminated planes is never a shade line.

Fig. 35.



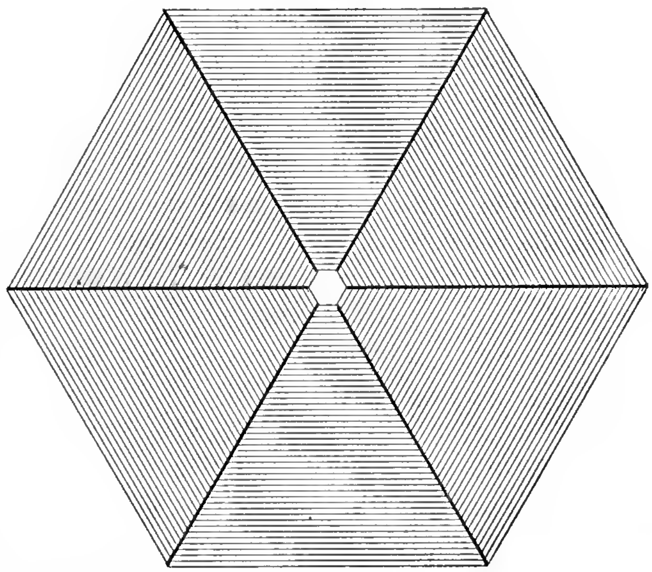

68. The conic section called the *parabola* furnishes another interesting exercise in ruled lines, when it is represented by its tangents as in Fig. 35. The angle $CAE$ may be assumed at pleasure, and on the finished drawing the numbers may

be omitted, being given here merely to show the law of construction. All the divisions are equal, and like numbers are joined.

Some interesting mathematical properties of the curve will be found in Chapter V.

69. A pleasing design that will test the beginner's skill is that of Fig. 36. It is suggestive of a cobweb, and a skillful free-hand draughtsman could make it more realistic by adding the spider. Use the 60° triangle for the heavy diagonals and parallels to them; the T-rule for the horizontals. Pencil the diagonals first but ink them last.

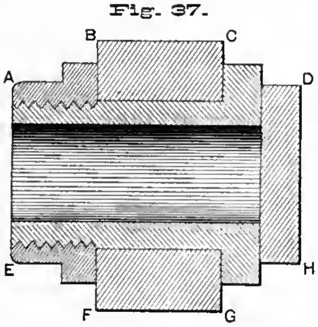
Fig. 36.



70. The even or flat effect of equidistant parallel lines is called *line-tinting;* or, if representing an object that has been cut by a plane, as in Fig. 37, it is called *section-lining.*

The *section,* strictly speaking, is the part actually in contact with the cutting plane; while the drawing as a whole is a *sectional view,* as it also shows what is back of the plane of section, the latter being always assumed to be transparent.

Fig. 37.

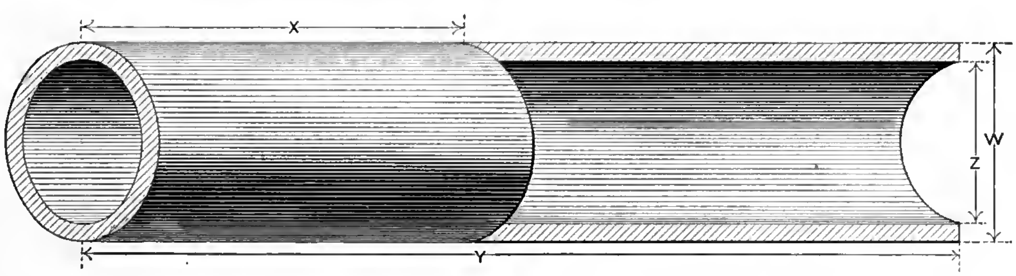



Adjacent pieces have the lines drawn in different directions in order to distinguish sufficiently between them.

The curved effect on the semi-cylinder is evidently obtained by properly varying both the strength of the line and the spacing.

71. The difference between the shading on the exterior and interior of a cylinder is sharply contrasted in Fig. 38. On the concavity the darkest line is at the top, while on the convex surface it is near the bottom, and below it the *spaces* remain unchanged while the lines diminish. A better effect would have been obtained in the figure had the engraver begun to increase the lines with the first decrease in the space between them.

Fig. 38.




The spacing of the lines, in section-lining, depends upon the scale of the drawing. It may run down to a thirtieth of an inch or as high as one-eighth; but from a twentieth to a twelfth of an inch would be best adapted to the ordinary range of work. *Equal* spacing and not fine spacing

should be the object, and neither scale nor patent section-liner should be employed, but distances gauged by the eye alone.

72. A refinement in execution which adds considerably to the effect is to leave a white line between the top and left-hand outlines of each piece and the section lines. When purposing to produce this effect rule light pencil lines as limits for the line-tints.

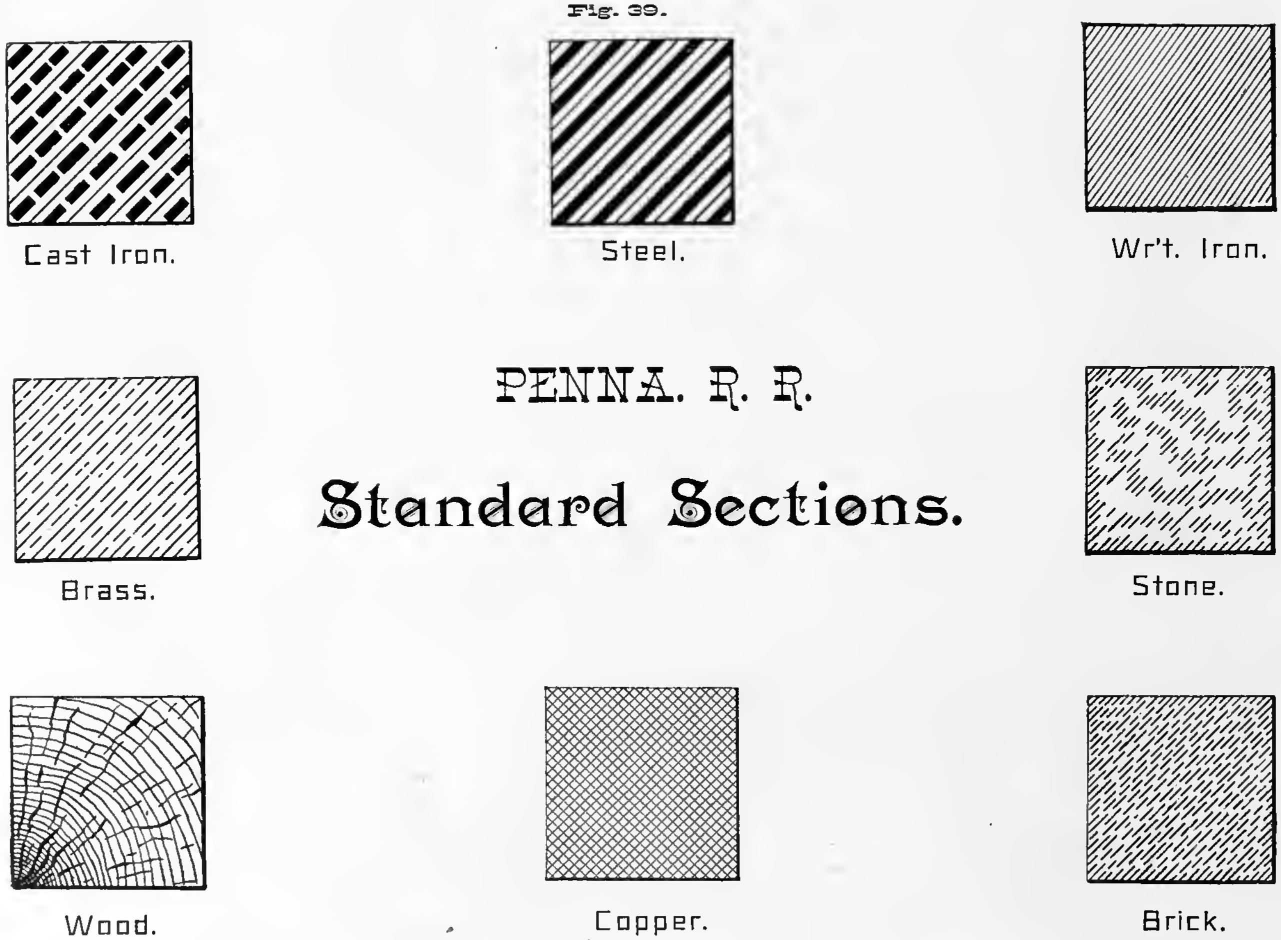
73. If the various pieces shown in a section are of different materials there are four ways of denoting the difference between them:

(a) By the use of the brush and certain water-colors, a method considerably employed in Europe, but not used to any great extent in this country, probably owing to the fact that it is not applicable where blue-prints of the original are desired.

The use of colors may, however, be advantageously adopted when making a highly finished, shaded drawing; the shading being done first, in India ink or sepia, and then overlaid with a flat tint of the conventional color. The colors ordinarily used for the metals are

| | | |
|---|---|---|
| Payne's gray or India ink | for | Cast Iron. |
| Gamboge | " | Brass (outside view). |
| Carmine | " | Brass (in section). |
| Prussian Blue | " | Wrought Iron. |
| Prussian Blue with a tinge of Carmine | " | Steel. |

Fig. 39.



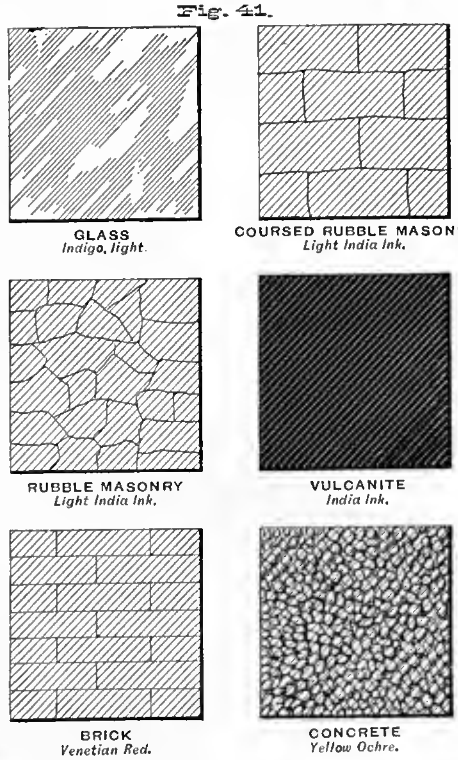
More natural effects can also be given by the use of colors, in representing the other materials of construction; and the more of an artist the draughtsman proves to be the closer can he approximate to nature.

Pale blue may be used for *water lines;* Burnt Sienna, whether grained or not, suggests *wood;* Burnt Umber is ordinarily employed for *earth;* either Light Red or Venetian Red are well adapted for *brick*, and a wash of India ink having a tinge of blue gives a fair suggestion of *masonry;* although the actual tint and surface of any rock can be exactly represented after a little practice with the brush and colors. These points will be enlarged upon later.

(b) By section-lining with the drawing pen in the conventional colors just mentioned, a process giving very handsome and thoroughly intelligible results on the original drawing, but, as before, unadapted to blue-printing and therefore not as often used as either of the following methods.

(c) By section-lining uniformly in *ink* throughout and printing the name of the material upon each piece.

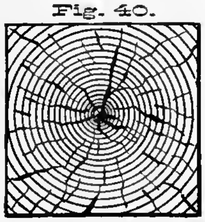
Fig. 41.




(d) By alternating light and heavy, continuous and broken lines according to some law. Said "law" is, unfortunately, by no means universal, despite the attempt made at a recent convention of the American Society of Mechanical Engineers to secure uniformity. Each draughting office seems at present to be a law unto itself in this matter.

74. As affording valuable examples for further exercise with the ruling pen the system of section-lines adopted by the Pennsylvania Railroad is presented on the opposite page. The wood section is an exception to the rule, being drawn free-hand, with a Falcon pen.

Fig. 40.



By way of contrasting free-hand with mechanical work Fig. 40 is introduced, in which the rings showing annual growth are drawn as concentric circles with the compass.

In Fig. 41 a few other sections appear, selected from the designs of M. N. Forney and F. Van Vleck, and which are fortunate arrangements.

75. Figs. 42 and 43 are profiles or outlines of mouldings, such as are of frequent occurrence in architectural work. It is good practice to convert such views into oblique projections, giving the effect of solidity; and to further bring out their form by line shading. Figs. 44–46 are such representations, the front of each being of the same *form* as Fig. 42. The oblique lines are all parallel to each other, and—where

Fig. 42.

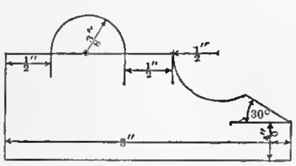


Fig. 43.

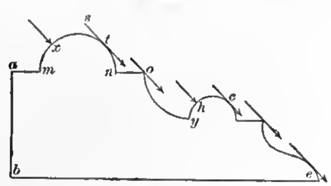
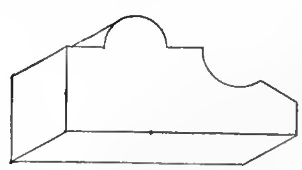


Fig. 44.



visible throughout—of the same length. Their *direction* should be chosen with reference to best exhibiting the peculiar features of the object. Obviously the view in Fig. 44 is the least adapted to the conveying of a clear idea of the moulding, while that of Fig. 46 is evidently the best.

76. The student may, to advantage, design profiles for mouldings and line-shade them, after converting them into oblique views. As hints for such work two figures are given (47–48), taken

Fig. 45.

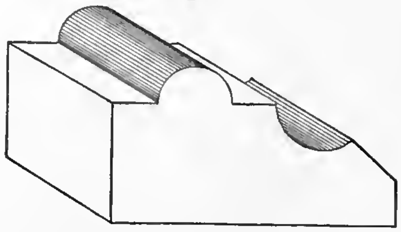
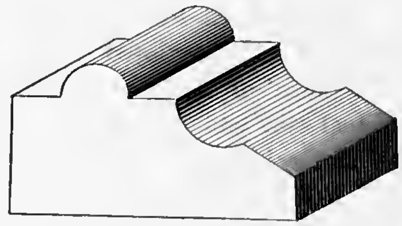


Fig. 46.



from actual construction in wood. By setting a moulding vertically, as in Fig. 49, and projecting horizontally from its points, a front view is obtained, as in Fig. 50.

Fig. 47.

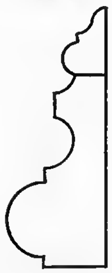


Fig. 48.

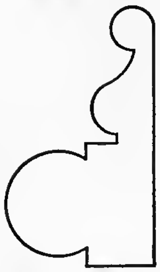


Fig. 49.

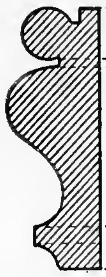
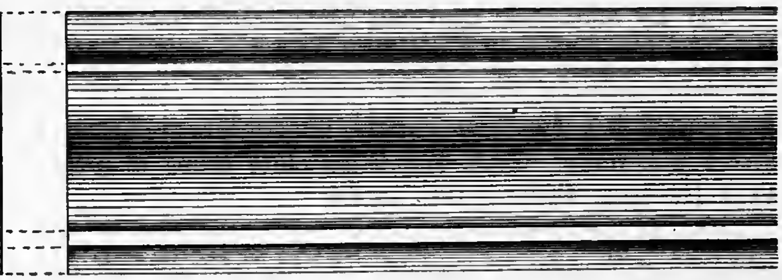
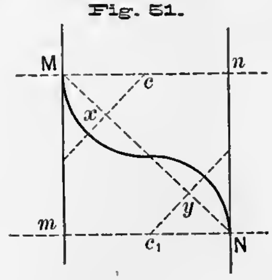


Fig. 50.



77. The *reverse curves* on the mouldings may be drawn with the irregular curve, (see Art 58); or, if composed of circular arcs to be tangent to vertical lines, by the following construction:—

Fig. 51.



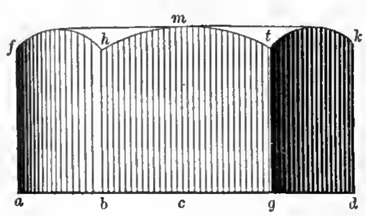

Let $M$ and $N$ be the points of tangency on the verticals $Mm$ and $Nn$, and let the arcs be tangent to each other at the middle point of the line $MN$. Draw $Mn$ and $Nm$ perpendicular to the vertical lines. The centres, $c$ and $c_1$, of the desired arcs, are at the intersection of $Mn$ and $Nm$ by perpendiculars to $MN$ from $x$ and $y$, the middle points of the segments of $MN$.

78. The light is to be assumed as coming in the usual direction, i. e., descending from left to right at such an angle that any ray would be projected on the paper at an angle of 45° to the horizontal.

In Fig. 43 several rays are shown. At $x$, where the light strikes the cylindrical portion most directly—technically is *normal* to the surface—is actually the brightest part. A tangent ray $st$ gives $t$, the darkest part of the cylinder. The concave portion beginning at $o$ would be darkest at $o$ and get lighter as it approaches $y$.

Flat parts are either to be left white, if in the light, or have equidistant lines if in the shade, unless the most elegant finish is desired, in which case both change of space and gradation of line must be resorted to as in Fig. 52, which represents a front view of a hexagonal nut. The front face, being parallel to the paper, receives an even tint. An inclined face *in the light*, as $abhf$, is lightest toward the observer, while an unilluminated face $tkdg$ is exactly the reverse.

Fig. 52.




Notice that to give a *flat* effect on the inclined faces the spacing-out as also the change in the size of lines must be more gradual than when indicating curvature. (Compare with Figs. 46 and 50.)

If two or more illuminated flat surfaces are parallel to the paper (as $t\,g\,b\,h$, Fig. 52) but at different distances from the eye, the nearest is to be the lightest; if unilluminated, the reverse would be the case.

79. In treating of the theory of shadows distinctions have to be made, not necessary here, between *real* and *apparent* brilliant points and lines. We may also remark at this point that to an experienced draughtsman some license is always accorded, and that he can not be expected to adhere rigidly to theory when it involves a sacrifice of effect. For example, in Fig. 46 we are unable to see to the left of the (theoretically) lightest part of the cylinder, and find it, therefore, advisable to move the darkest part past the point where, according to Fig. 43, we know it in reality to be. The professional draughtsmen who draw for the best scientific papers and to illustrate the circulars of the leading machine designers allow themselves the latitude mentioned, with most pleasing results. Yet until one may be justly called an expert he can depart but little from the narrow confines of theory without being in danger of producing decidedly peculiar effects.

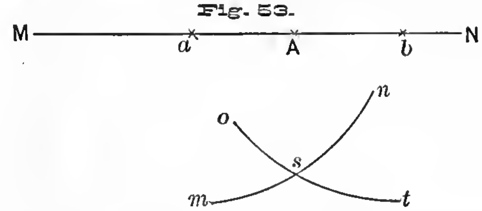
80. As from this point the student will make considerable use of the compasses, a few of the more important and frequently recurring plane problems, nearly all of which involve their use, may well be introduced. The proofs of the geometrical constructions are in several cases omitted, but if desired the student can readily obtain them by reference to any synthetic geometry or work on plane problems.

All the problems given (except No. 20) have proved of value in shop practice and architectural work.

The student should again read Arts. 48–51 regarding special uses of the 30° and 45° triangles, which, with the T-rule, enable him to employ so many "draughtsman's" as distinguished from "geometrician's" methods; also Arts. 36 and 37.

81. *Prob. 1. To draw a perpendicular to a given line at a given point,* as $A$ (Fig. 53), use the triangles, or triangle and rule as previously described; or lay off equal distances $A\,a$, $A\,b$, and with $a$ and $b$ as centres draw arcs $o\,s\,t$, $m\,s\,n$, with common radius greater than one-half $a\,b$. The required perpendicular is the line joining $A$ with the intersection of these arcs.

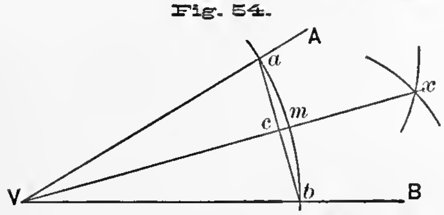
Fig. 53.




82. *Prob. 2. To bisect a line,* as $M\,N$, use its *extremities* exactly as $a$ and $b$ were employed in the preceding construction, getting also a second pair of arcs (same radius for all the arcs) intersecting *above* the line at a point we may call $x$. The line from $s$ to $x$ will be a bisecting perpendicular.

83. *Prob. 3. To bisect an angle,* as $A\,V\,B$, (Fig. 54), lay off on its sides *any* equal distances $V\,a$, $V\,b$. Use $a$ and $b$ as centres for intersecting arcs having a common radius. Join $V$ with $x$, the intersection of these arcs, for the bisector required.

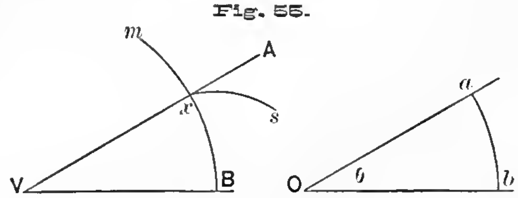
Fig. 54.




84. *Prob. 4. To bisect an arc of a circle,* as $a\,m\,b$ (Fig. 54), bisect the chord $a\,c\,b$ by Prob. 2; or, by Prob. 3, bisect the angle $a\,V\,b$ which subtends the arc.

85. *Prob. 5. To construct an angle equal to a given angle,* as $\theta$ (Fig. 55), draw *any* arc $a\,b$ with centre $O$, then, with same radius, an indefinite arc $m\,B$, centre $V$; use the *chord* of $a\,b$ as a radius, and from centre $B$ cut the arc $m\,B$ at $x$. Join $V$ and $x$. Then angle $A\,V\,B$ equals $\theta$.

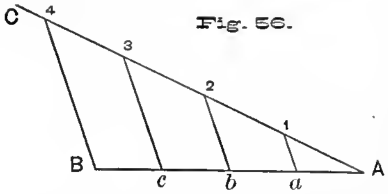
Fig. 55.




86. *Prob. 6. To pass a circle through three points,* $a$, $b$ and $c$, join them by lines $a\,b$, $b\,c$, bisect these lines by perpendiculars, and the intersection of the latter will be the centre of the desired circle.

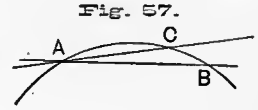
87. *Prob.* 7. *To divide a line into any number of equal parts* draw from one extremity as $A$ (Fig. 56) a line $AC$ making any random angle with the given line $AB$. With a scale point off on $AC$ as many *equal* parts (size immaterial) as are required on $AB$; four, for example. Join the last point of division (4) with $B$; then parallels to such line from the other points will divide $AB$ similarly.

Fig. 56.



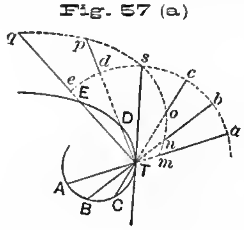

88. A *secant* to a curve is a line cutting it in two points. If the secant $AB$ be turned to the left about $A$ the point $B$ will approach $A$, and the line will pass through $AC$ and other secant positions. When $B$ reaches and coincides with $A$ the line is said to be *tangent* to the curve. (See also Art. 368.)

Fig. 57.




A *tangent to a mathematical curve* is determined by means of known properties of the curve. For a *random* or *graphical* curve the method illustrated by Fig. 57 (a) is the most accurate and is as follows: Through $T$, the point of desired tangency, draw random secants to points on either side of it, as $A$, $B$, $D$, etc., and prolong them to meet a circle having centre $T$ and *any* radius. On each secant lay off—from its intersection with the circle—the chord of that secant in the random curve. Thus, $am = TA$; $bn = TB$; $pd = TD$. From $s$, where the curve $m\,n\,o\,p\,q$ cuts the circle, draw $sT$, which will be a tangent, since for it the chord has its minimum value.

Fig. 57 (a)




A *normal* to a curve is a line perpendicular to the tangent, at the point of tangency. In a circle it coincides in direction with the radius to the point of tangency.

89. *Prob.* 8. *To draw a tangent to a circle at a given point* draw a radius to the point. The perpendicular to this radius at its extremity will be the required tangent. Solve with triangles.

90. *Prob.* 9. *To draw a tangent to a circle from a point without* join the centre $C$ (Fig. 58) with the given point $A$; describe a semicircle on $AC$ as a diameter and join $A$ with $D$, the intersection of the arcs. $ADC$ equals 90°, being inscribed in a semi-circle; $AD$ is then the required tangent, being perpendicular to $CD$ at its extremity.

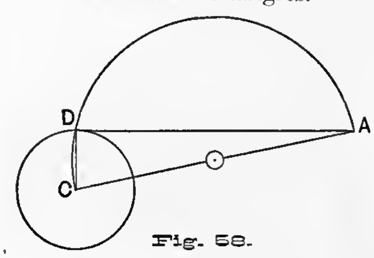
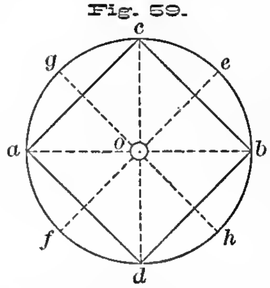



Fig. 58.

91. *Prob.* 10. *To draw a tangent at a given point of a circular arc whose centre is unknown or inaccessible*, locate *on the arc* two points equidistant from the given point and on opposite sides of it; the chord of these points will be parallel to the tangent sought.

92. A *regular* polygon has all its sides equal, as also its angles. If of three sides it is called the *equilateral triangle;* four sides, the *square;* five, *pentagon;* six, *hexagon;* seven, *heptagon;* eight, *octagon;* nine, *nonagon* or *enneagon;* ten, *decagon;* eleven, *undecagon;* twelve, *dodecagon.*

Fig. 59.

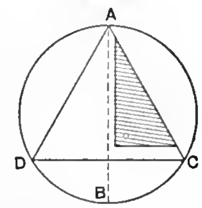



The angles of the more important regular polygons are as follows: *triangle*, 120°; *square*, 90°; *pentagon*, 72°; *hexagon*, 60°; *octagon*, 45°; *decagon*, 36°; *dodecagon*, 30°. The angle at the vertex of a regular polygon is the supplement of its central angle.

93. For the polygons most frequently occurring there are many special methods of construction. All but the pentagon and decagon can be readily inscribed or circumscribed about a circle by the use of the T-rule and triangle. For example, draw $ab$ (Fig. 59) with the T-rule, and $cd$ perpendicular to it with a triangle. The 45° triangle will then give a *square*, $acbd$. The same triangle in two positions would give $ef$ and $gh$, whence $ag$, $gc$, etc., would be sides of a regular *octagon.*

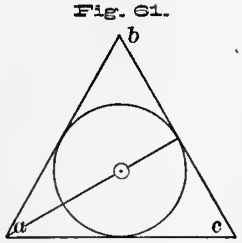
94. The 60° triangle used as in Art. 51 would give the *hexagon;* and alternate vertices of the latter, joined, would give an *equilateral triangle.* Or the radius of the circle stepped off six times on the circumference, and alternate points connected, would result similarly.

Fig. 60.




95. *Prob. 11.* An additional method for *inscribing an equilateral triangle in a circle, when one vertex of the triangle is given,* as *A,* Fig. 60, is to draw the diameter, *A B,* through *A,* and use the triangle to obtain the sides *A C* and *A D,* making angles of 30° with *A B.* *D* and *C* will then be the extremities of the third side of the triangle sought.

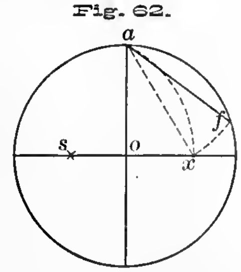
Fig. 61.




96. *Prob. 12.* *To inscribe a circle in an equilateral triangle* draw a perpendicular from any vertex to the opposite side. The centre of the circle will be on such line, two-thirds of the distance from vertex to base, while the radius desired will be the remaining third. (Fig. 61).

97. *Prob. 13.* *To inscribe a circle in any triangle* bisect any two of the interior angles. The intersection of these bisectors will be the *centre,* and its perpendicular distance from any side will be the *radius* of the circle sought.

Fig. 62.

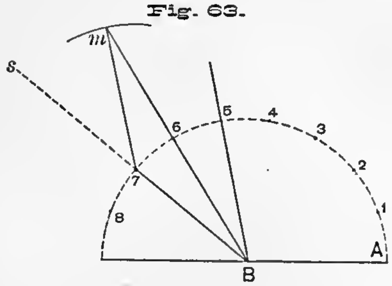



98. *Prob. 14.* *To inscribe a pentagon in a circle* draw mutually perpendicular diameters (Fig. 62); bisect a radius as at *s;* draw arc *a x* of radius *s a* and centre *s;* then chord $a\,x = a\,f$, the side of the pentagon to be constructed.

99. *Prob. 15.* *To construct a regular polygon of any number of sides, the length of the side being given.*

Let *A B* (Fig. 63) be the length assigned to a side, and a regular polygon of *x* sides desired. Take *x* equal to *nine* for illustration, draw a semi-circle with *A B* as radius, and divide by trial into *x* (or 9) equal parts. Join *B* with $x-2$ points of division, or *seven,* beginning at *A,* and prolong all but the last. With 7 as a centre, radius *A B,* cut line B–6 at *m* by an arc, and join *m* with 7, giving another side of the required polygon. Using *m* in turn as a centre, same radius as before, cut B–5 (produced) and so obtain a third vertex.

Fig. 63.




This solution is based on the familiar principles (a) that if a regular polygon has *x* sides each interior angle equals $\frac{180°\,(x-2)}{x}$, and (b) that the diagonals drawn from any vertex of the polygon make the same angles with each other as with the sides meeting at that vertex.

100. *Prob. 16.* *Another solution of Prob. 15.* Erect a perpendicular *H R* (Fig. 64) at the middle point of the given side. With *M* as a centre, radius *M S,* describe arc *S A* and divide it by trial into six equal parts. Arcs through these points of division, using *A* as a centre, and numbered up from six, give the centres on the vertical line for circles passing through *M* and *S* and in which *M S* would be a chord as many times as the number of the centre.

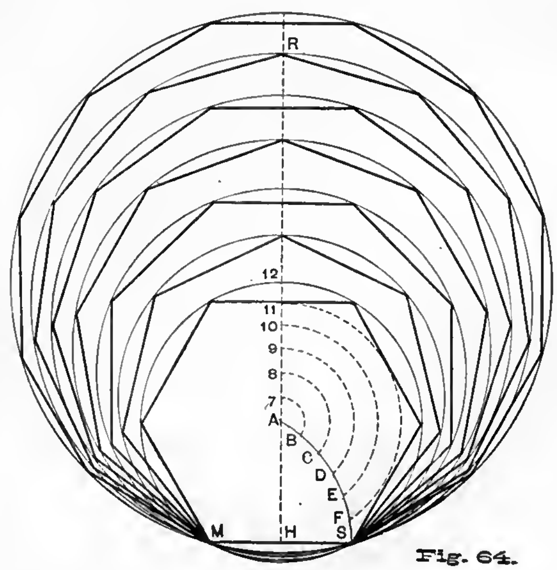



Fig. 64.

101. For any unusual number of sides the method

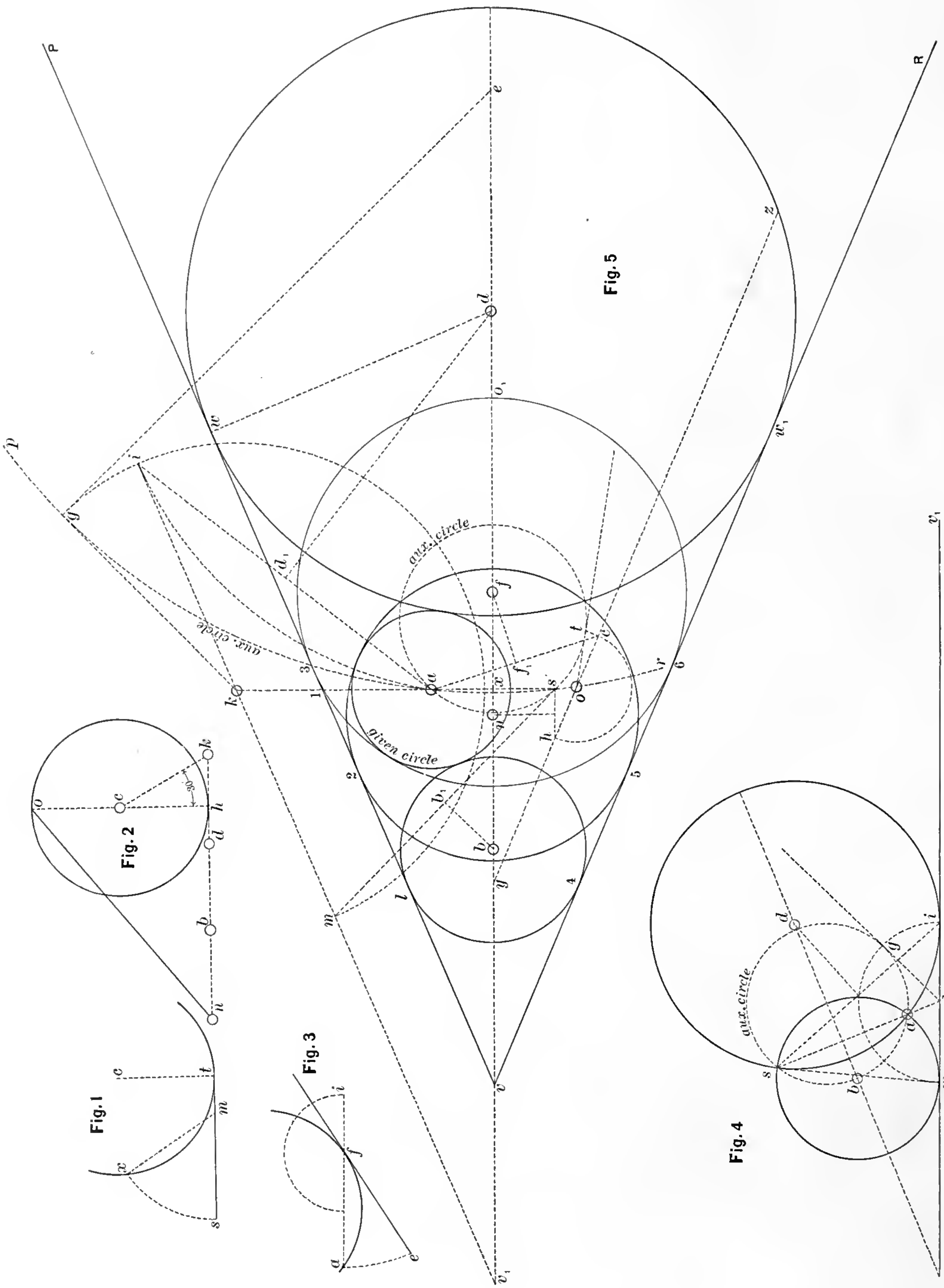


Fig.1
Fig.2
Fig.3
Fig.4
Fig.5
aux. circle
given circle

by "trial and error" is often resorted to, and even for ordinary cases it is by no means to be despised. By it the dividers are set "by guess" to the probable chord of the desired arc, and, supposing a *heptagon* wanted, the chord is stepped off seven times around the circumference; care being taken to have the points of the dividers come exactly *on* the arc, and also to avoid damaging the paper. If the seventh step goes past the starting point the dividers require closing; if it falls short, the original estimate was evidently too small. Obviously the change in setting the dividers ought in this case to be, as nearly as possible, *one-seventh* of the error; and after a few trials one should "come out even" on the last step.

102. *Prob. 17. To lay off on a given circle an arc of the same length as a given straight line.*[1] Let $t$ (Plate I, Fig. 1) be one extremity of the desired arc; $ts$ the given straight line and tangent to the circle; $tm$ equal one-fourth of $ts$, and $sx$ drawn with centre $m$, radius $ms$. Then the length of the arc $tx$ is a close approximation to that of the line $ts$.

103. *Prob. 18. To lay off on a straight line the length of a given circular arc,*[1] or, technically, to *rectify* the arc, let $af$ (Plate I, Fig. 3) be the given arc; $ai$ the chord prolonged till $fi$ equals one-half the chord $af$; and $ae$ an arc drawn with radius $ai$, centre $i$. Then $fe$ approximates closely to the length of the arc $af$.

104. *Prob. 19. To obtain a straight line equal in length to any given semi-circle,*[2] draw a diameter $oh$ of the given semi-circle (Plate I, Fig. 2) and a radius inclined at an angle of 30° to the radius $ch$. Prolong the radius to meet the line $bhk$, drawn tangent to the circle at $h$. From $k$ lay off the radius three times, reaching $n$. The line $no$ equals the semi-circumference to four places of decimals.

105. *Prob. 20. To draw a circle tangent to two straight lines and a given circle.* (Four solutions.) This problem is given more on account of the valuable exercise it will prove to the student in absolute precision of construction than for its probable practical applications. Fig. 4 (Plate I) illustrates the geometrical principles involved, and in it a circle is required to contain the points $s$ and

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[1] These methods of approximation were devised by Prof. Rankine. They are sufficiently accurate for arcs not exceeding 60°. The error varies as the fourth power of the angle. The complete demonstration of Prob. 17 can be found in the Philosophical Magazine for October, 1867, and of Prob. 18 in the November issue of the same year.

[2] In his *Graphical Statics* Cremona states this to be the simplest method known for rectifying a semi-circumference. According to Böttcher it is due to a Polish Jesuit, Kochansky, and was published in the Acta Eruditorum Lipsiae, 1685. The demonstration is as follows: Calling the radius *unity*, the diameter would have the numerical value 2.

Then in Fig. 2, Plate I, we have $on = \sqrt{oh^2 + hn^2} = \sqrt{oh^2 + (kn - kh)^2} = \sqrt{4 + (3 - \tan 30°)^2} = 3.14159 +$

The tangent of an angle (abbreviated to "tan.") is a trigonometric function whose numerical value can be obtained from a table. A draughtsman has such frequent occasion to use these functions that they are given here for reference, both as lines and as ratios.

*Trigonometric Functions as Ratios.*

$\theta$ = the given angle = $CAB$
$h$ = hypothenuse of triangle $CAB$
$a$ = $AB$ = side of triangle *adjacent* to vertex of $\theta$
$o$ = $BC$ = side of triangle *opposite* to $\theta$

Then $\sin\theta = \frac{o}{h}$; $\cos\theta = \frac{a}{h}$;

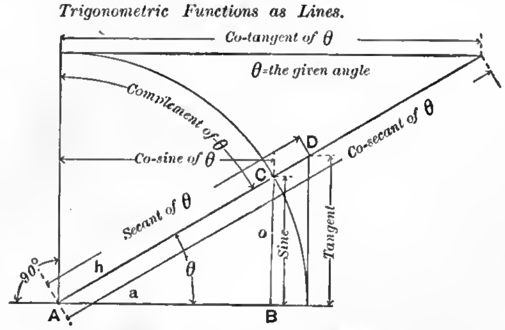
$\tan\theta = \frac{o}{a} = \frac{\sin\theta}{\cos\theta}$;

$\sec\theta = \frac{h}{a}$ = reciprocal of cosine.

$\operatorname{cosec}\theta = \frac{h}{o}$ " " sine

$\operatorname{cotan}\theta = \frac{a}{o} = \frac{\cos\theta}{\sin\theta}$ = reciprocal of tan $\theta$.

*Trigonometric Functions as Lines.*




The prefix "co" suggests "complement;" the *co*-sine of $\theta$ is the sine of the complement of $\theta$, &c. As *lines* the functions may be defined as follows:

The *sine* of an arc (e. g., that subtended by angle $\theta$ in the figure) is the perpendicular ($CB$) let fall from one extremity of the arc upon the diameter passing through the other extremity. If the radius A C, through one extremity of the arc, be prolonged to cut a line tangent at the other extremity, the intercepted portion of the tangent is called the *tangent* of the arc, and the distance, on such extended radius, from the centre of the circle to the tangent, is called the *secant of the arc.*

The co-sine, co-secant and co-tangent of the arc are respectively the sine, secant and tangent of the complement of the given arc.

$a$ and be tangent to the line $m\,v_1$. Draw first *any* circle containing $s$ and $a$, as the one called "aux. circle." Join $s$ to $a$ and prolong to meet $m\,v_1$ at $k$. From $k$ draw a tangent, $k\,g$, to the auxiliary circle. With radius $k\,g$ obtain $m$ and $i$ on the line $m\,v$. A circle through $s$, $a$ and $m$, or through $s$, $a$ and $i$ will fulfill the conditions. For $k\,g^2 = k\,s \times k\,a$, as $k\,g$ is a tangent and $k\,s$ a secant. But $k\,i = k\,g$, therefore $k\,i^2 = k\,s \times k\,a$, which makes $k\,i$ a tangent to a circle through $s$, $a$ and $i$.

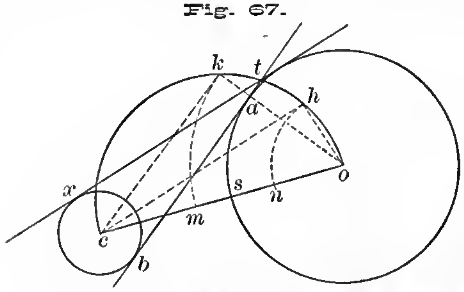
In Fig. 5 (Plate I) the construction is closely analogous to the above, and the lettering identical for the first half of the work. The "given circle" is so called in the figure; the given lines are $P\,v$ and $R\,v$. Having drawn the bisector, $v\,e$, of the angle $P\,v\,R$, locate $s$ as much below $v\,e$ as $a$ (the centre of the given circle) is above it, the line $a\,s$ being perpendicular to $v\,e$. Draw $v_1\,m\,k\,i$ parallel to $v\,p$ and at a distance from it equal to the *radius* of the given circle. Then $s$, $a$, $k$ and $m\,v_1$ of Fig. 5 are treated exactly as the analogous points of Fig. 4, and a circle obtained (centre $d$) containing $a$, $s$ and $i$. The required circle will have the same centre $d$ but radius $d\,w$, shorter than the first by the distance $w\,i$. Treat $s$, $a$, and $m$, (Fig. 5), similarly, getting the smallest of the four possible circles.

The remaining solutions are obtained by using the points $a$ and $s$ again, but in connection with a line $y\,z$ parallel to $v\,R$ and *inside* the angle, again at a perpendicular distance from one of the given lines equal to the radius of the given circle.*

This problem makes a handsome plate if the *given* and *required* lines are drawn in *black;* the lines giving the first two solutions in *red;* the remaining construction lines in *blue.*

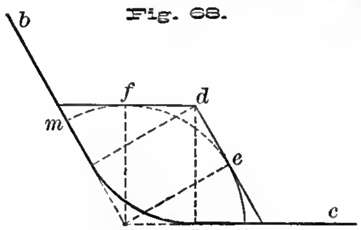
106. *Prob. 21. To draw a tangent to two given circles* (a problem that may occur in connecting band-wheels by belts) join their centres, $c$ and $o$, (Fig. 67) and at $s$ lay off $s\,m$ and $s\,n$ each equal to the radius of the smaller circle. Describe a semi-circle $o\,h\,k\,c$ on $o\,c$ as a diameter. Carry $m$ and $n$ to $k$ and $h$, about $o$ as a centre. Angles $c\,k\,o$ and $c\,h\,o$ are each $90°$, being inscribed in a semi-circle; and $c\,k$ is parallel to $a\,b$, which last is one of the required tangents; while $c\,h$ is parallel to $t\,x$, a second tangent. Two more can be similarly found.

Fig. 67.



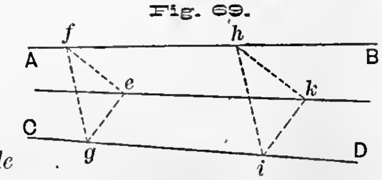

107. *Prob. 22. To unite two inclined straight lines by an arc tangent to both, radius given.* Prolong the given lines to meet at $a$ (Fig. 68). With $a$ as a centre and the given radius describe the arc $m\,n$. Parallels to the given lines and tangent to arc $m\,n$ meet at $d$, from which perpendiculars to the given lines give the points of tangency of the required arc, which is now drawn with the given radius.

Fig. 68.




108. *Prob. 23. To draw through a given point a line which will—if produced—pass through the inaccesssible*

Fig. 69.

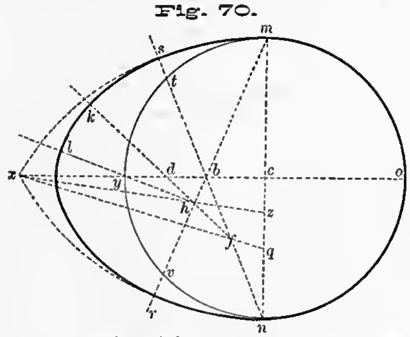



*This solution is taken from Benjamin Alvord's *Tangencies of Circles and of Spheres*, published by the *Smithsonian Institute*. That valuable pamphlet presents geometrical solutions of the ten problems of Apollonius on the tangencies of circles, and also of the fifteen problems on the tangencies of spheres, all of which are valuable to the draughtsman, both geometrically and as exercises in precision. The solutions are based on the principle, illustrated by Fig. 57, that the tangent line or tangent curve is the limit of all secant lines or curves. The problems on the tangencies of circles are as follows, the number of solutions in each case being given: (1) To draw a circle through three points. One solution. (2) Circle through two points and tangent to a given straight line. Two solutions. (3) Circle through a given point and tangent to two straight lines. Two solutions. (4) Circle through two points and tangent to a given circle. Two solutions. (5) Circle through a given point, tangent to a given straight line and a given circle. Four solutions. (6) Circle through a given point and tangent to two given circles. Four solutions. (7) Circle tangent to three straight lines, two only of which may be parallel. Four solutions. (8) Circle tangent to two straight lines and a given circle. Four solutions. (Art. 105, above). (9) Circle tangent to two given circles and a given straight line. Eight solutions. (10) Circle tangent to three given circles. Eight solutions.

*intersection of two lines.* Join the given point $e$ with *any* point $f$ on $A\ B$, and also with some point $g$ on $C\ D$. From any point $h$ on $A\ B$ draw $h\ i$ parallel to $f\ g$, then $i\ k$ parallel to $g\ e$, and $h\ k$ parallel to $f\ e$. The line $k\ e$ will fulfill the conditions.

109. *Prob. 24. To draw an oval upon a given line.* Describe a circle on the given line, $m\ n$, (Fig. 70) as a diameter. With $m$ and $n$ as centres describe arcs, $m\ x$, $n\ x$, radius $m\ n$. Draw $m\ v$ and $n\ t$ through $v$ and $t$, the middle points of the quadrants $y\ m$, $y\ n$. Then $m\ s$ and $r\ n$ are the portions of $m\ x$ and $n\ x$ forming part of the oval. Bisect $n\ c$ at $q$ and draw $q\ x$. Also bisect $c\ q$ at $z$ and join the latter with $x$. Bisect $y\ b$ in $d$ and draw $f\ d$ from $f$, the intersection of $n\ s$ and $q\ x$. Use $f$ as a centre, and $f\ s$ as radius, for an arc $s\ k$ terminating on $f\ d$. The intersection, $h$, of $k\ f$ with $x\ z$, is then the next centre, and $h\ k$ the radius of the arc $k\ l$ which terminates on $h\ y$ produced. The oval is then completed with $y$ as a centre and radius $y\ l$. The lower portion is symmetrical with the upper and therefore similarly constructed.

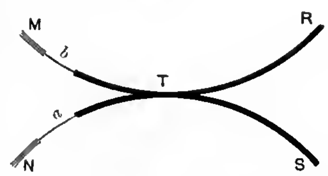
Fig. 70.




110. Where exact tangency is the requirement novices occasionally endeavor to conceal a failure to secure the desired object by thickening the curve. Such a course usually defeats itself and makes more evident the error they thus hope to conceal. With such instruments of precision as the draughtsman employs there can be but little excuse, if any, either for overlapping or falling short.

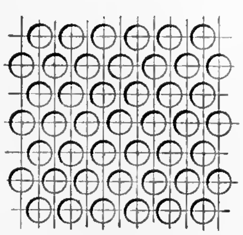
A common error in drawing tangents, where the lines are of appreciable thickness, is to make the *outsides* of the lines touch; whereas they should have their thickness in common at the point of tangency, as at $T$ (Fig. 70), where, evidently, the centre-lines $a$ and $b$ of the arcs would be *exactly* tangent, while the outer arc of $M$ would come tangent to the inner arc of $N$, and vice versa.

Fig. 71.




111. When either a tube or a solid cylindrical piece is seen in the direction of its axis the outline is, obviously, simply a circle; and often the only way to determine which of the two the circle represented would be to notice which part of said end view was represented as casting the shadow. In Fig. 72, if the shaded arcs can cast shadows, the space *inside* the circles must be open, and the figure would represent a portion of the end view of a boiler with its tubular openings.

Fig. 72.



By exactly reversing the shading (the effect of which can be seen by turning the figure upside down) it is converted into a drawing of a number of solid, cylindrical pins projecting from a plate.

The tapering begins at the extremities of a diameter drawn at 45° to the horizontal.

*To get a perfect taper on small circles* use the bow-pen, and, after making one complete circle, add the extra thickness by a second turn, which is to begin with the pen-point *in the air*, the pen being brought down gradually upon the paper, and then, while turning, raised from it again.

*On medium and large circles* the requisite taper can be obtained by a different process, viz., by using the same radius again but by taking a *second centre*, distant from the first by an amount equal to the proposed width of the broadest part of the shaded arc; the line through the two centres to be perpendicular to that diameter which passes through the extremities of the taper. The extra thickness should be inside the circumference, not outside.

112. As exercises in concentric circles Figs. 73 and 74 will prove a good test of skill. They represent, either entire or in section, a gymnasium ring, the "annular torus" of mathematical works.

Fig. 73.

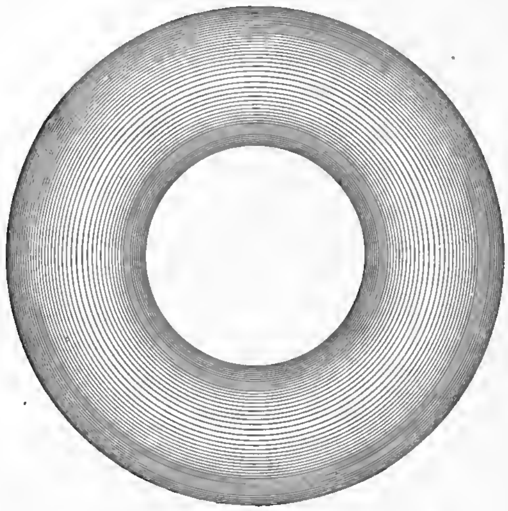
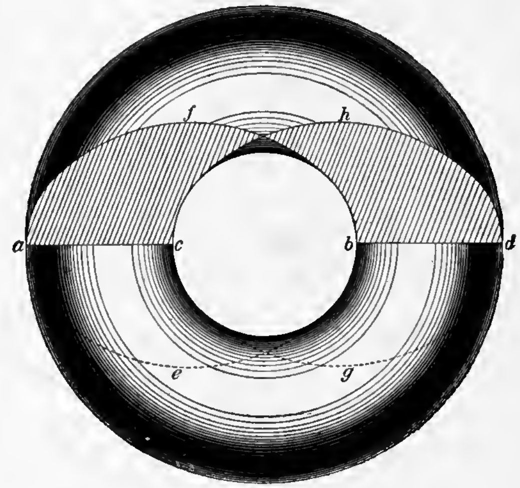
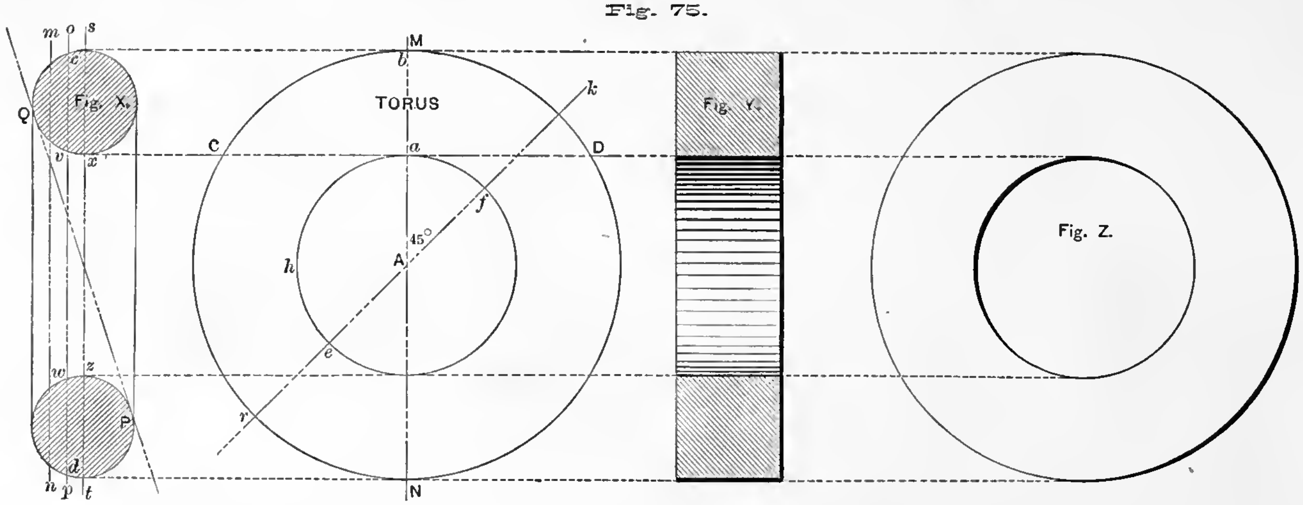


Fig. 74.




It is a surface possessing some remarkable properties, chief among which is the fact that it is the only surface of revolution known from which two circles can be cut by each plane in three different systems of planes.* In two of these systems each plane will cut two *equal* circles from the surface.

Fig. 75.




113. In Fig. 75 the same surface is shown in front view, between sub-figures X and Y. The axis of the surface will be perpendicular to the paper at $A$. If $MN$ represents a plane perpendicular to the paper and containing the axis, then Fig. X will show the shape of the cut or *section*. As $MN$ was but one of the positions of a plane containing the axis, and as the surface might be generated by rotating $MN$ with the circle $ab$ about the axis, it is evident that *in one of the three systems of planes mentioned in the last article each plane must contain the axis.*

When a surface can be generated by revolution about an axis one of its characteristics is that *any plane perpendicular to the axis will cut it in a circle.* The circles of Fig. 73 may then be, for the moment, considered as parallel cuts by a series of planes perpendicular to the axis, a few of which

* Olivier, *Memoires de Géométrie Descriptive.* Paris, 1851.

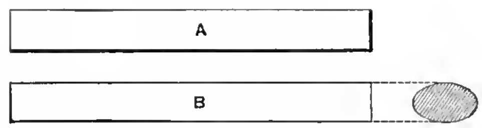
may be shown in $m\,n$, $o\,p$, &c. (Fig. X). Each of these planes cuts two circles from the surface; the plane $o\,p$, for example, giving circles of diameters $c\,d$ and $v\,w$ respectively.

A plane, perpendicular to the paper on the line $P\,Q$, would be a *bi-tangent* plane, because tangent to the surface at two points, $P$ and $Q$; and such plane would cut *two* over-lapping circles from the torus, each of them running partly on the inner and partly on the outer portion of the surface. These sections are seen as ellipses in Fig. 74. For the proof that such sections are circles the student is probably not prepared at this point, but is referred to *Olivier's Seventh Memoir*, or to the appendix.*

114. Another interesting fact with regard to the torus is that a series of planes *parallel to, but not containing the axis*, cut it in a set of curves called the Cassian ovals (see Art. 212) of which the Lemniscate of Art. 158 is a special case, and which would result from using a plane parallel to the axis and tangent to the surface at a point on the smallest circle at $a$, (Fig. 75.)†

115. Fig. Y is given to illustrate the fact that from mere untapered outlines, such as compose the central figure, we cannot determine the form of the object. By shading $e\,h\,f$ and $D\,N\,r$ we get Fig. Z, and the form shown in Fig. Y would be instantly recognized without the drawing of the latter. An angular object must therefore have shade lines, as also the *end view* of a round object; but a side view of a cylindrical piece must either have *uniform outlines* or be shaded with several lines.

Fig. 76.




Thus, in Fig. 76, $A$ would represent an angular piece, while $B$ would indicate a circular cylinder; if *elliptical* its section would be drawn at one side as shown.

116. Before presenting the crucial test for the learner—the railroad rail—two additional practice exercises, mainly in ruling, are given in Figs. 77 and 78. The former shows that, like the parabola the circle and hyperbola can be represented by their enveloping tangents. The upper and lower figures are merely two views of the surface called the *warped hyperboloid*, from the hyperbolas which constitute the curved outlines seen in the upper figure. The student can make this surface in a few moments by stringing threads through equidistant holes arranged in a circle on two circular discs of the same or different sizes, but having the *same number of holes in each disc.* By attaching weights to the threads to keep them in tension at all times, and giving the upper disc a twist, the surface will change from cylindrical or conical to the hyperboloidal form shown.

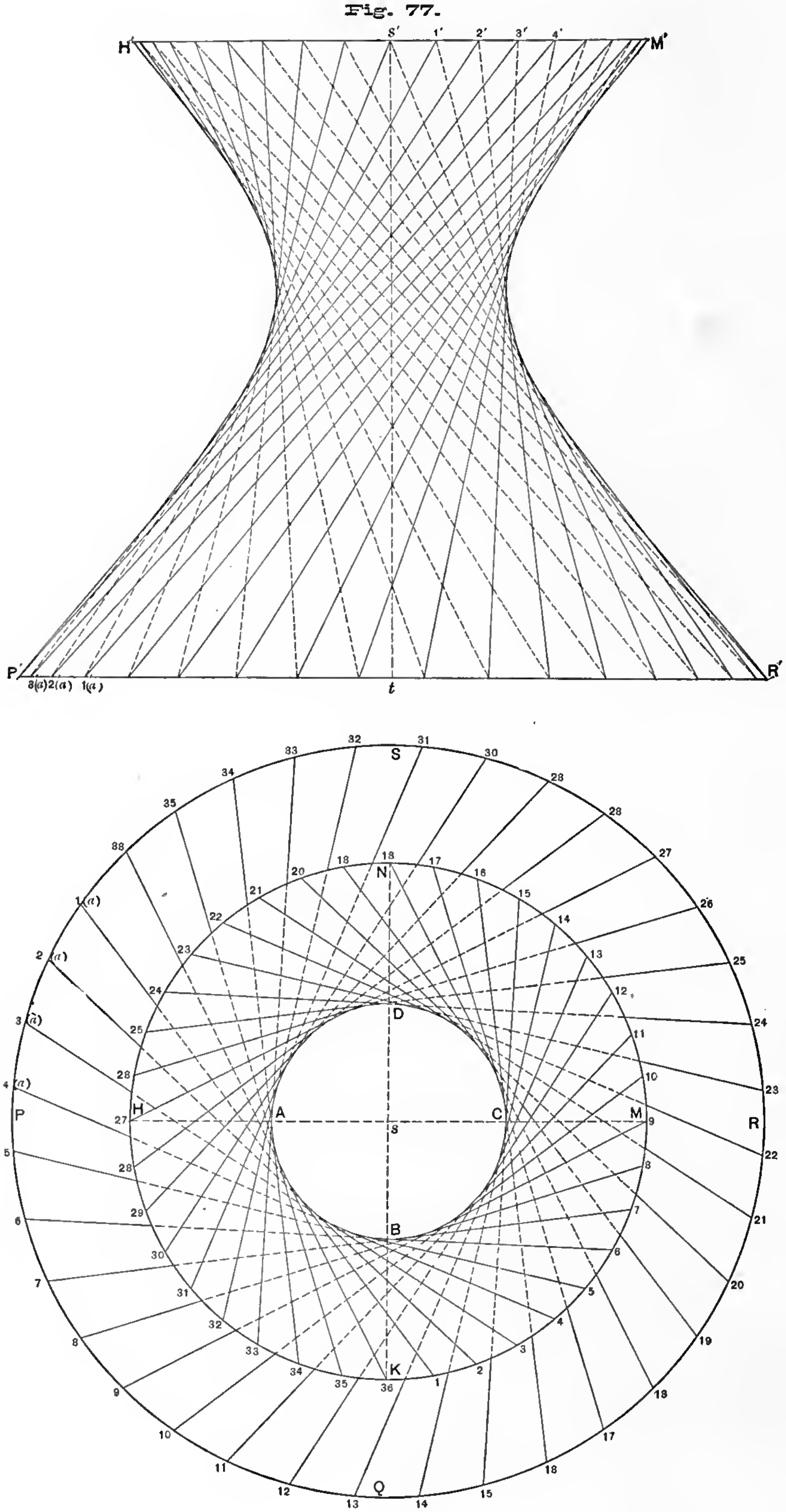
Gear wheels are occasionally constructed, having their teeth upon such a surface and in the direction of the lines or elements forming it; but the hyperboloid is of more interest mathematically than mechanically.

Begin the drawing by pencilling the three concentric circles of the lower figure. When inking, omit the smaller circle. Draw a series of tangents to the inner circle, each one beginning on the middle circle and terminating on the outer. Assume any vertical height, $t\,s'$, for the upper figure, and draw $H'\,M'$ and $P'\,R'$ as its upper and lower limits. $H'\,M'$ is the vertical projection, or elevation, of the circle $H\,K\,M\,N$, and *all* points on the latter, as $1, 2, 3, 4$, are projected, by perpendiculars to $H'\,M'$, at $1', 2', 3', 4'$, etc. All points on the larger circle $P\,Q\,R$ are similarly projected to $P'\,R'$. The extremities of the same tangent are then joined in the upper view, as $1'$ with $1$ $(a)$.

* An original demonstration by Mr. George E. Barton (Princeton, '95,) when a Junior in the John C. Green School of Science.

† These curves can also be obtained by assuming two foci, as if for an ellipse, but taking the *product* of the focal radii as a constant quantity, some perfect square. If $pp' = 36$ then a point on the curve would be found at the intersection of arcs having the foci as centres, and for radii $2''$ and $18''$, or $4''$ and $9''$, etc. The Lemniscate results when the constant assumed is the square of half the distance between the foci.

Fig. 77.

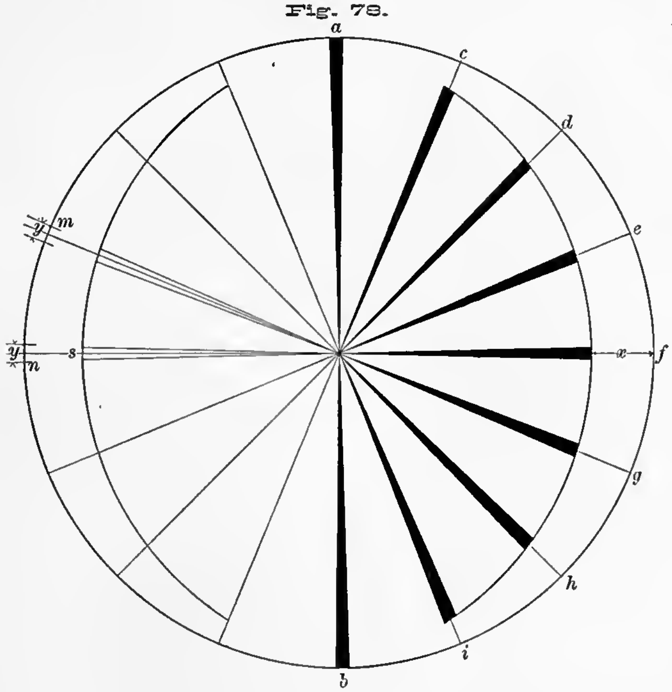


Part of each line is dotted to represent its disappearing upon an invisible portion of the surface. The law of such change on the lower figure is evident from inspection, while on the elevation the point of division on each line is exactly above the point where the other view of the same line runs through $HM$ in the lower figure.

**117.** To reproduce Fig. 78 draw first the circle $afbn$, then two circular arcs which would contain $a$ and $b$ if extended, and whose greatest distance from the original circle is $x$, (arbitrary). Sixteen equidistant radii as at $a, c, d$, etc., are next in order, of which the rule and 45° triangle give those through $a, d, f$ and $h$. At their extremities, as $m$ and $n$, lay off the desired width, $y$, and draw toward the points thus determined lines radiating from the centre. Terminate these last upon the inner arcs. Ink by drawing *from* the centre, *not through* or *toward it.*

All construction lines should be erased before the tapering lines are filled in. The "filling in" may be done very rapidly by ruling the edges in *fine* lines at first, then opening the pen slightly and beginning again where the opening between the lines is apparent and ruling from there, adding thickness to each edge on its inner side. It will then be but a moment's work to fill in, freehand, with the Falcon pen or a fine-pointed sable-brush, between the—now heavy—edge-lines of the taper. To have the pen make a coarse line when starting from the centre would destroy the effect desired.

Fig. 78.




118. The draughtsman's ability can scarcely be put to a severer test on mere outline work than in the drawing of a railroad rail, so many are the changes of radii involved.

As previously stated, where tangencies to straight lines are required, *the arcs are to be drawn first*, then the tangents.

Figs. 79 and 80 are photo-engravings of rail sections, showing two kinds of "finish." Fig. 80 is a "working drawing" of a Pennsylvania Railroad rail, full-size. This makes one of the handsomest plates that can be undertaken, if finished with shade lines, as in Fig. 79, section-lined with Prussian blue, and the dimension lines drawn in carmine.

A still higher effect is shown in the woodcut on page 85, the rail being represented in oblique projection and shaded.

Begin Fig. 80 by drawing the vertical centre-line, it being an axis of symmetry. Upon it lay off 5″ for the total height, and locate two points between the top and base, at distances from them of $1\frac{3}{4}''$ and $\frac{7}{8}''$ respectively; these to be the points of convergence of the lower lines of the head and sloping sides of the base. From these points draw lines, at first indefinite in length, and inclined 13° to the horizontal. The top of the head is an arc of 10″ radius, subtended by an angle of 90°. This changes into an arc of $\frac{7}{16}''$ radius on the upper corner, with its centre on the side of said 9° angle. The sides of the head are straight lines, drawn at 4° to the vertical, and tangent to the corner arcs. The thin vertical portion of the rail is called the *web*, and is $\frac{17}{32}''$ wide at its centre. The outlines of the web are arcs of 8″ radius, subtended by angles of 15°, centres on line marked "centre line of bolt holes."

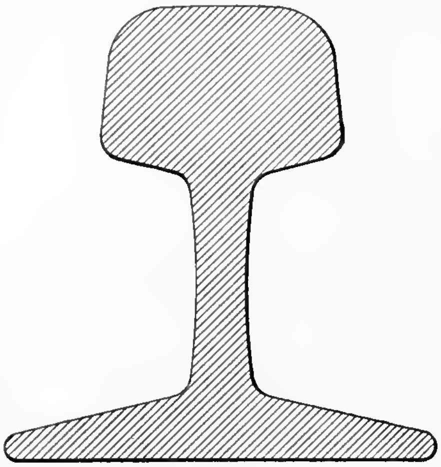
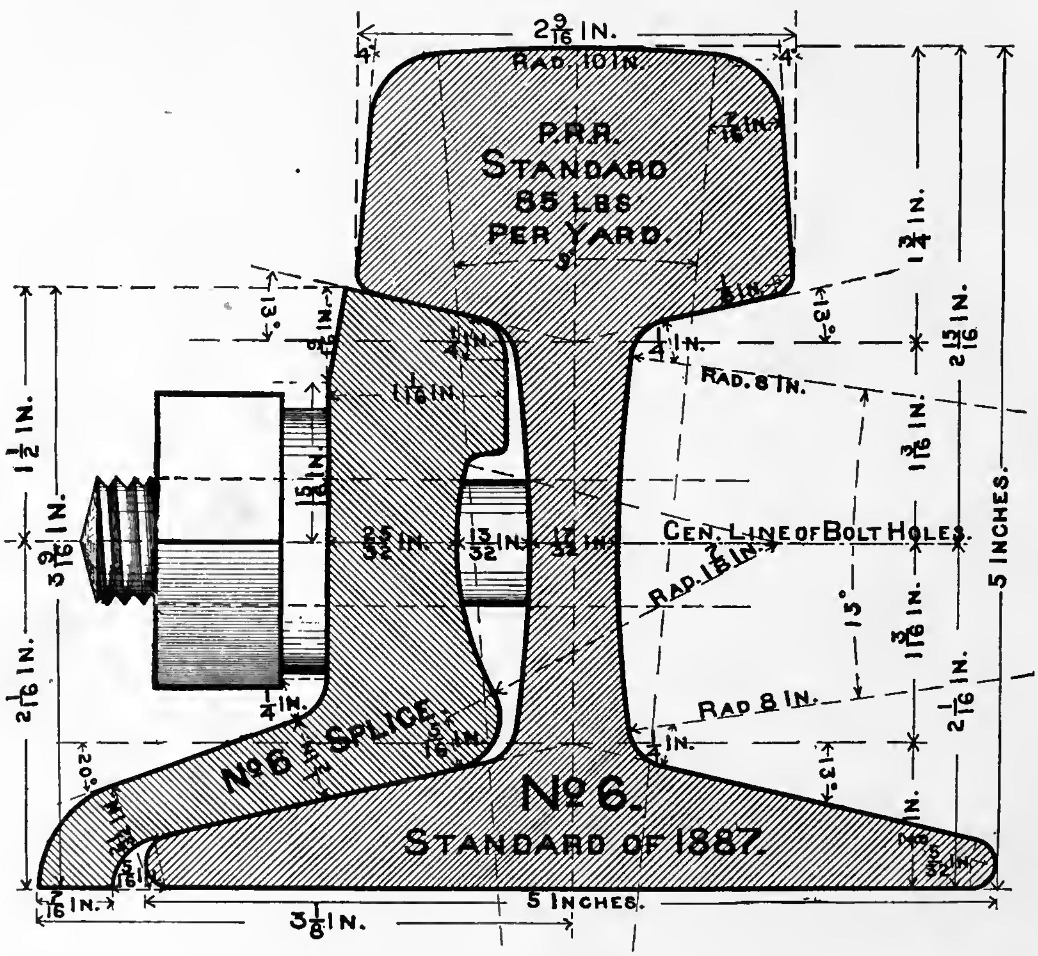


Fig. 79.

The weight per yard of the rail shown is given as eighty-five pounds,* from which we know the area of the cross-section to be eight and one-half square inches, since a bar of iron a yard long and one square inch in cross-section weighs, approximately, ten pounds. (10.2 lbs., average).

The proportions given are slightly different from those recommended in the report† of the committee appointed by the American Society of Civil Engineers to examine into the proper relations to each other of the sections of railway wheels and rails. There was quite general agreement as to the following recommendations: a top radius of twelve inches; a quarter-inch corner radius; vertical sides to the web; a lower-corner of one-sixteenth inch, and a broad head relatively to the depth.

Fig. 80.




* See the Appendix for dimensions of a 100-lb. rail.

† Transactions A. S. C. E., January, 1891.

## *CHAPTER V.*

THE HELIX.—CONIC SECTIONS.—HOMOLOGICAL PLANE CURVES AND SPACE-FIGURES.—LINK-MOTION CURVES.—CENTROIDS.—THE CYCLOID.—COMPANION TO THE CYCLOID.—THE CURTATE TROCHOID.—THE PROLATE TROCHOID.—HYPO-, EPI-, AND PERI-TROCHOIDS.—SPECIAL TROCHOIDS—ELLIPSE, STRAIGHT LINE, LIMAÇON, CARDIOID, TRISECTRIX, INVOLUTE, SPIRAL OF ARCHIMEDES.—PARALLEL CURVES.—CONCHOID.—QUADRATRIX.—CISSOID.—TRACTRIX.—WITCH OF AGNESI.—CARTESIAN OVALS.—CASSIAN OVALS.—CATENARY.—LOGARITHMIC SPIRAL.—HYPERBOLIC SPIRAL.—THE LITUUS.—THE IONIC VOLUTE.

119. There are many curves which the draughtsman has frequent occasion to make whose construction involves the use of the irregular curve. The more important of these are the Helix; Conic Sections—Ellipse, Parabola and Hyperbola; Link-motion curves or point-paths; Centroids; Trochoids; the Involute and the Spiral of Archimedes. Of less practical importance, though equally interesting geometrically, are the other curves mentioned in the heading.

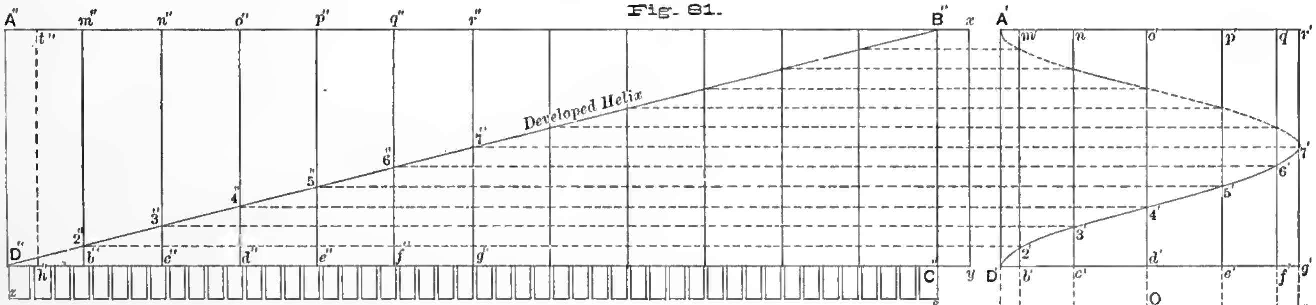
The student should become thoroughly acquainted with the more important geometrical properties of these curves, both to facilitate their construction under the varying conditions that may arise and also as a matter of education. Considerable space is therefore allotted to them here.

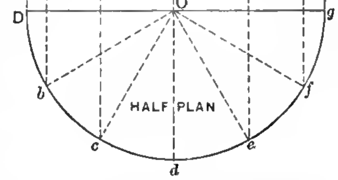
At this point Art. 58 should be reviewed, and in addition to its suggestions the student is further advised to work, at first, on as large a scale as possible, not undertaking small curves of sharp curvature until after acquiring some facility with the curved ruler.

### THE HELIX.

120. The ordinary helix is a curve which cuts all the elements of a right cylinder at the same angle. Or we may define it as the curve which would be generated by a point having a uniform motion around a straight line combined with a uniform motion parallel to the line.

Fig. 81.




The student can readily make a model of the cylinder and helix by drawing on thick paper or Bristol-board a rectangle $A''B''C''D''$ (Fig. 81) and its diagonal, $D''B''$; also equidistant elements, as $m''b''$, $n''c''$, etc. Allow at the right and bottom about a quarter of an inch extra for overlapping, as shown by the lines $xy$ and $sz$. Cut out the rectangle $zx$; also cut a series of vertical slits between $D''C''$ and $zs$; put mucilage between $B''C''$ and $xy$; then roll the paper up into cylindrical form, bringing $A''D''t''h''$ in front of and upon the gummed portion, so that $A''D''$

will coincide with $B'' C''$. The diagonal $D'' B''$ will then be a helix on the outside of the cylinder, but half of which is visible in front view, as $D' 7'$, (see right-hand figure); the other half, $7' A'$, being indicated as unseen.

To give the cylinder permanent form it can then be pasted to a cardboard base by mucilage on the under side of the marginal flaps below $D'' C''$, turning them *outward*, not in toward the axis.

The rectangle $A'' B'' C'' D''$ is called the *development* of the cylinder; and any surface like a cylinder or cone, which can be rolled out on a plane surface and its equivalent area obtained by bringing consecutive elements into the same plane, is called a *developable surface*. The elements $m'' b''$, $n'' c''$, etc., of the development stand vertically at $b, c, d \dots g$ of the half plan, and are seen in the elevation at $m' b'$, $n' c'$, $o' d'$, etc. The point $3'$, where any element, as $c'$, cuts the helix, is evidently as high as $3''$, where the same point appears on the development. We may therefore get the curve $D' 7' A'$ by erecting verticals from $b, c, d \dots g$, to meet horizontals from the points where the diagonal $D'' B''$ crosses those elements on the development. $D'' C''$ obviously equals $2 \pi r$, where $r = O D$.

*The shortest method of drawing a helix* is to divide its *plan* (a circle) and its *pitch* ($D' A'$, the *rise* in one turn) into the same number of equal parts; then verticals $b m'$, $c n'$, etc., from the points of division on the plan, will meet the horizontals dividing the pitch, in points $2'$, $3'$, etc., of the desired curve.

The construction of the helix is involved in the designing of screws and screw-propellers, and in the building of winding stairs and skew-arches.

Mathematically, both the curve and its orthographic projection are well worth study, the latter being always a *sinusoid*, and becoming the *companion to the cycloid* for a 45°-helix. (Arts. 170 and 171).

For the *conical helix*, seen in projection and development as a Spiral of Archimedes, see Art. 191.

## THE CONIC SECTIONS.

121. The ellipse, parabola and hyperbola are called *conic sections* or *conics* because they may be obtained by cutting a cone by a plane. We will, however, first obtain them by other methods.

According to the definition given by Boscovich, the ellipse, parabola and hyperbola are curves in which there is a *constant ratio* between the distances of points on the curve from a certain fixed point (the *focus*) and their distances from a fixed straight line (the *directrix*).

Referring to the parabola, Fig. 82, if $S$ and $B$ are points of the curve, $F$ the focus and $X Y$ the directrix, then, if $S F : S T :: B F : B X$, we conclude that $B$ and $S$ are points of a conic section.

122. The actual value of such ratio (or *eccentricity*) may be 1 or either greater or less than unity. When $S F$ equals $S T$ the ratio equals 1, and the relation is that of equality, or parity, which suggests the *parabola*.

123. If it is farther from a point of the conic to the focus than to the directrix the ratio is greater than 1, and the *hyperbola* is indicated.

124. The *ellipse*, of course, comes in for the third possibility as to ratio, viz., less than 1. Its construction by this principle is not shown in Fig. 82 but later, the (Art. 142) method of generation here given illustrating the practical way in which, in landscape gardening, an elliptical plat would be laid out; it is therefore called the construction as the "gardener's ellipse."

Taking $A C$ and $D E$ as representing the extreme length and width, the points $F$ and $F_1$ (*foci*) would be found by cutting $A C$ by an arc of radius equal to one-half $A C$, centre $D$. Pegs or pins at $F$ and $F_1$, and a string, of length $A C$, with ends fastened at the foci, complete the preliminaries. The curve is then traced on the ground by sliding a pointed stake against the string, as at $P$, so that at all times the parts $F_1 P$, $F P$, are kept straight.

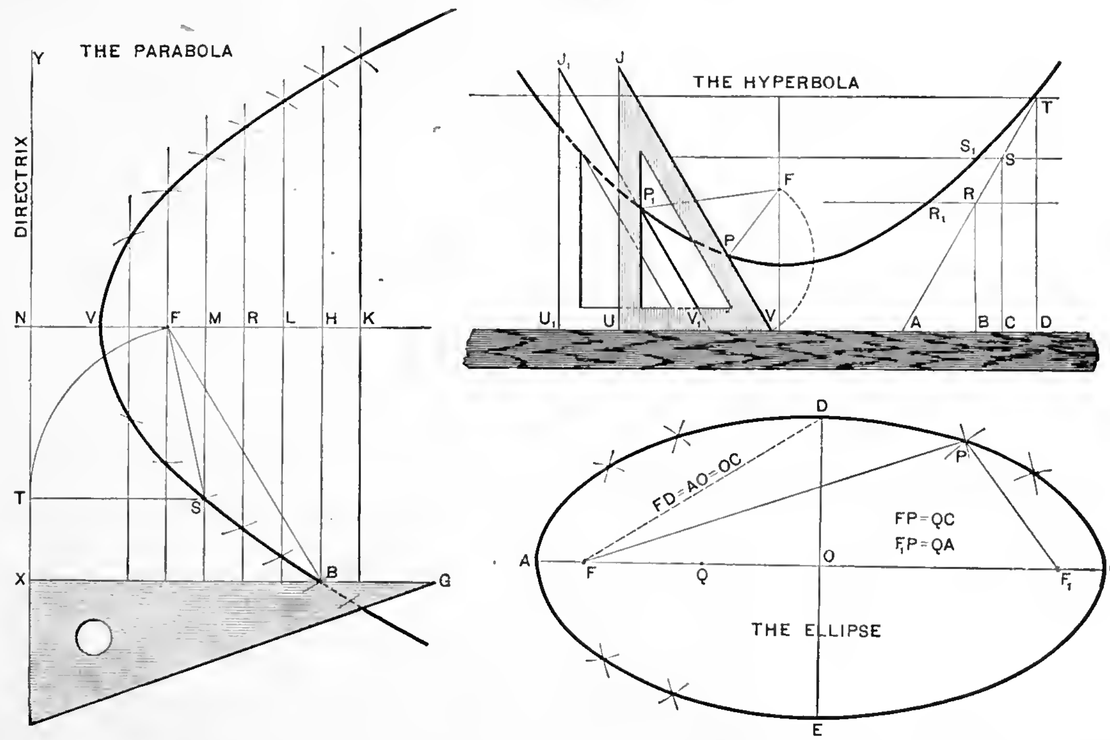
125. According to the foregoing construction the ellipse may be defined as a curve in which *the sum of the distances from any point of the curve to two fixed points is constant.* That *constant* is evidently the longer or *transverse* (*major*) axis, $AC$. The shorter or *conjugate* (or *minor*) axis, $DE$, is perpendicular to the other.

With the compasses we can determine $P$ and other points of the ellipse by using $F$ and $F_1$ as centres, and for radii any two segments of $AC$. $Q$, for example, gives $AQ$ and $CQ$ as segments. Then arcs from $F$ and $F_1$, with radius equal to $QC$, would intersect arcs from the same centres, radius $QA$, in four points of the ellipse, one of which is $P$.

126. By the Boscovich definition we are also enabled to construct the parabola and hyperbola by continuous motion along a string.

For the *parabola* place a triangle as in Fig. 82, with its altitude $GX$ toward the focus. If a string *of length* $GX$ be fastened at $G$, stretched tight from $G$ to *any* point $B$, by putting a pencil at $B$, then the remainder $BX$ swung around and the end fastened at $F$, it is then, evidently, as far from $B$ to $F$ as it is from $B$ to the directrix; and that relation will remain constant as the triangle is slid along the directrix, if the pencil point remains against the edge of the triangle so that the portion of the string from $G$ to the pencil is kept straight.

Fig. 82.




127. For the *hyperbola*, (Fig. 82), the construction is identical with the preceding, except that the string fastened at $J$ runs down the *hypothenuse*, and equals it in length.

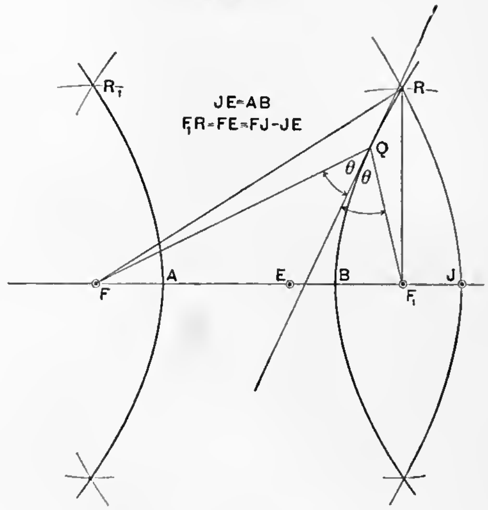
128. Referring back to Fig. 35, it will be noticed that the focus and directrix of the parabola are there omitted; but the former would be the point of intersection of a perpendicular from $A$ upon the line joining $C$ with $E$. A line through $A$, parallel to $CE$, would be the directrix.

129. Like the ellipse, the hyperbola can be constructed by using two foci, but whereas in the ellipse (Fig. 82) it was the *sum* of two focal radii that was constant, i. e., $FP + F_1P = FD +$

$F_1 D = A C$ (the transverse axis), it is the *difference* of the radii that is constant for the hyperbola.

In Fig. 83 let $A B$ be the transverse axis of the two arcs, or "branches," which make the complete hyperbola; then using $\rho$ and $\rho'$ to represent *any* two focal radii, as $F Q$ and $F_1 Q$, or $F R$ and $F_1 R$, we will have $\rho - \rho' = A B$, the constant quantity.

Fig. 83.




To get a point of the curve in accordance with this principle we may lay off from either focus, as $F$, any distance greater than $F B$, as $F J$, and with it as a radius, and $F$ as a centre, describe the indefinite arc $J R$. Subtracting the constant, $A B$, from $F J$, by making $J E = A B$, we use the remainder, $F E$, as a radius, and $F_1$ as a centre, to cut the first arc at $R$. The same radii will evidently determine three other points fulfilling the conditions.

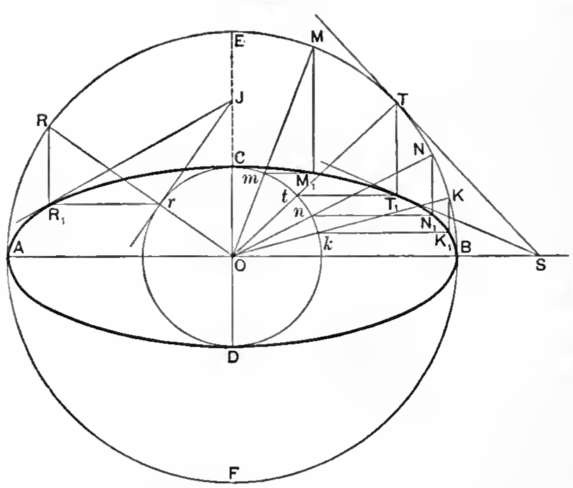
130. *The tangent to an hyperbola* at any point, as $Q$, bisects the angle $F Q F_1$, between the focal radii.

In the *ellipse*, (Fig. 82), it is the *external* angle between the radii that is bisected by the tangent.

In the *parabola*, (Fig. 82), the same principle applies, but as one focus is supposed to be *at infinity*, the focal radius, $B G$, toward the latter, from any point, as $B$, would be parallel to the axis. The tangent at $B$ would therefore bisect the angle $F B X$.

131. *The ellipse as a circle viewed obliquely.* If $A R M B F$ (Fig. 84) were a circular disc and we were to rotate it on the diameter $A B$, it would become narrower in the direction $F E$ until, if sufficiently turned, only an edge view of the disc would be obtained. The axis of rotation $A B$ would, however, still appear of its original length. In the rotation supposed, all points not on the axis would describe circles about it with their planes perpendicular to it. $M$, for example, would describe an arc, part of which is shown in $M M_1$, which is straight, as the plane of the arc is seen "edge-wise." If instead of a circular disc we turn an *elliptical* one, $A C B D$, upon its shorter axis $C D$, it is obvious that $B$ would apparently approach $O$ on one side while $A$ advanced on the other, and that the disc could reach a position in which it would be projected in the small circle $C k D$. If, then, the axes of an ellipse are given, as $A B$ and $C D$, use them as diameters of concentric circles; from their centre, $O$, draw random radii, as $O T$, $O K$; then either, as $O T$, will cut the circles in points, $t$ and $T$, through which a parallel and perpendicular, respectively, to the longer axis will give a point $T_1$ of an ellipse.

Fig. 84.



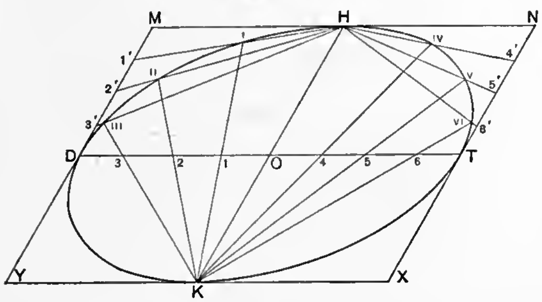

The relation just illustrated is established analytically in the Appendix.

132. If $T S$ is a tangent at $T$ to the large circle, then when $T$ has rotated to $T_1$ we shall have $T_1 S$ as a tangent to the ellipse at the point derived from $T$, the point $S$ having remained constant, being on the axis of rotation.

Similarly, if a tangent at $R_1$ were wanted, we would first find $r$, corresponding to $R_1$; draw the tangent $rJ$ to the small circle; then join $R_1$ to $J$, the latter on the axis and therefore constant.

133. Occasionally we have given the length and inclination of a pair of diameters of the ellipse making oblique angles with each other. Such diameters, are called *conjugate*, and the curve may be constructed upon them thus: Draw the axes $TD$ and $HK$ at the assigned angle $DOH$; construct the parallelogram $MNXY$; divide $DM$ and $DO$ into the same number of equal parts; then from $K$ draw lines through the points of division on $DO$, to meet similar lines drawn through $H$ and the divisions on $DM$. The intersection of like-numbered lines will give points of the ellipse.

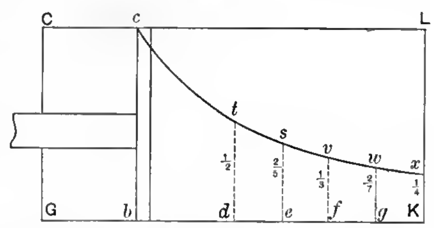
Fig. 85.




134. It is the law of expansion of a perfect gas that the volume is inversely as the pressure. That is, if the volume be doubled the pressure drops one-half; if trebled the pressure becomes one-third, etc. Steam not being a perfect gas departs somewhat from the above law, but the curve indicating the fall in pressure due to its expansion is compared with that for a perfect gas.

To construct the curve for the latter let us suppose $CLKG$ (Fig. 86) to be a cylinder with a volume of gas $CGbc$ behind the piston. Let $cb$ indicate the pressure before expansion begins. If the piston be forced ahead by the expanding gas until the volume is doubled, the pressure will drop, by Boyle's law, to one-half, and will be indicated by $td$. For three volumes the pressure becomes $rf$, etc. The curve $csx$ is an *hyperbola*, of the form called *equilateral*, or *rectangular*.

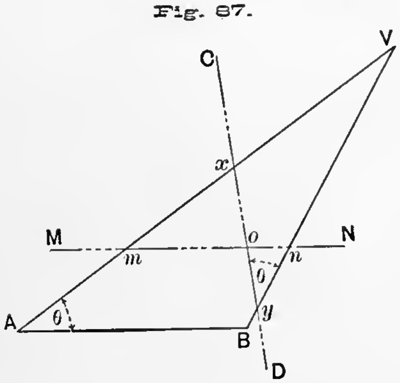
Fig. 86.




Suppose the cylinder were infinite in length. Since we cannot conceive a volume so great that it could not be doubled, or a pressure so small that it could not be halved, it is evident that theoretically the curve $ex$ and the line $GK$ will forever approach each other yet never meet; that is, they will be *tangent at infinity*. In such a case the straight line is called an *asymptote* to the curve.

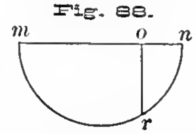
135. Although the *right* cone (i. e., one having its axis perpendicular to the plane of its base) is usually employed in obtaining the ellipse, hyperbola and parabola, yet the same kind of sections can be cut from an oblique or *scalene* cone of circular base, as $VAB$, Fig. 87. Two sets of *circular* sections can also be cut from such a cone, one set, obviously, by planes parallel to the base, while the other would be by planes like $CD$, making the same angle with the lowest element, $VB$, that the highest element, $VA$, makes with the base. The latter sections are called *sub-contrary*. Their planes are perpendicular to the plane $VAB$ containing the highest and lowest elements—*principal plane*, as it is termed.

Fig. 87.




To prove that the sub-contrary section $xy$ is a *circle* we note that both it and the section $mn$—the latter known to be a circle because parallel to the base—intersect in a line perpendicular to the paper at $o$. This line pierces the front surface of the cone at a point we may call $r$. It would be seen as the ordinate $or$ (Fig. 88), were the front half of the circle $mn$ rotated until parallel to the paper. Then $or^2 = om \times on$. But in Fig. 87 we have $om:oy::ox:on$, whence $oy \times ox = om \times on = or^2$, proving the section $xy$ circular.

Fig. 88.



Were the vertex of a scalene cone removed *to infinity* the cone would become an oblique cylinder with circular base; but the latter would possess the property just established for the former.

136. The most interesting practical application of the sub-contrary section is in Stereographic Projection, one of the methods of representing the earth's surface on a map. The especial convenience of this projection is due to the fact that in it every circle is projected as a circle. This results from the relative position of the eye (or centre of projection) and the plane of projection; the latter is that of some

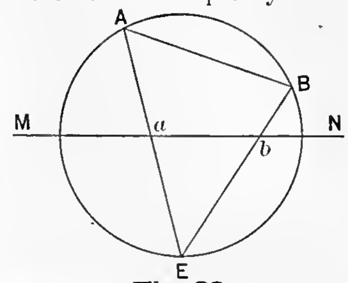



Fig. 89.

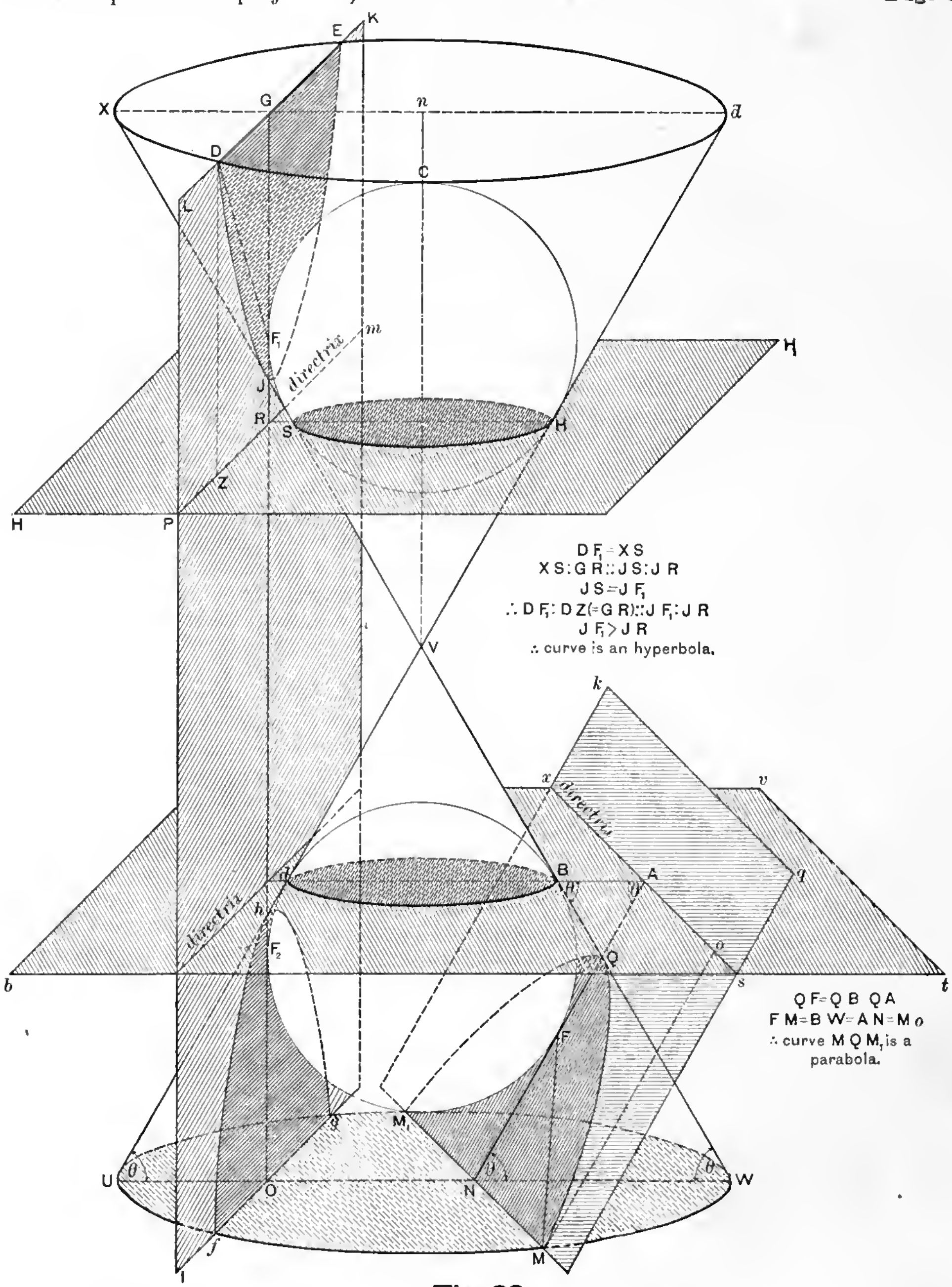



Fig. 90.

great circle of the earth, and the centre of projection is located at the pole of such circle.

137. In Fig. 89 let the circle $ABE$ represent the equator; $MN$ the plane of a meridian, also taken as the plane of projection; $AB$ any circle of the sphere; $E$ the position of the eye: then $ab$, the projection of $AB$ on plane $MN$, is a circle, being a sub-contrary section of the cone $E.AB$.

138. We now take up the conic sections as derived from a right cone.

A complete cone (Fig. 90) lies as much above as below the vertex. To use the term adopted from the French, it has two *nappes*.

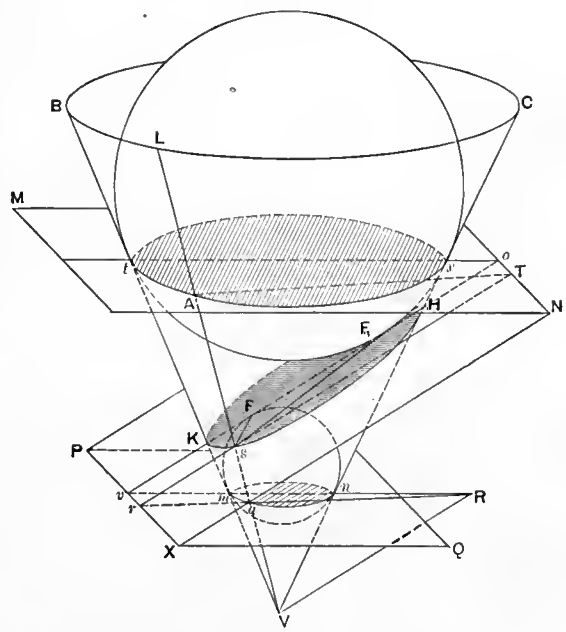
Aside from the extreme cases of *perpendicularity to* or *containing* the axis, *the inclination of a plane cutting the cone may be*

(a) *Equal* to that of the *elements* (see the remark in Art. 4), therefore parallel to *one* element, giving the *parabola*, as $MQM_1$ (Fig. 90); the plane $kqM$ being parallel to the element $VU$ and therefore making with the base the same angle, $\theta$, as the latter.

(b) *Greater than* that of the elements, causing the plane to cut both nappes and giving a two-branch curve, the *hyperbola*, as $DJE$ and $fhg$ (Fig. 90).

(c) *Less than* that of the elements, the plane therefore cutting all the elements on one side of the vertex, giving a closed curve, the ellipse; as $KsH$, Fig. 91.

Fig 91.




139. Figures 90 and 91, with No. 4 of Plate 2, are not only stimulating examples for the draughtsman, but they illustrate probably the most interesting fact met with in the geometrical treatment of conics, viz., that the spheres which are tangent simultaneously to the cone and the cutting planes, touch the latter in the foci of the conics; while in each case the directrix of the curve is the line of intersection of the cutting plane and the plane of the circle of tangency of cone and sphere.

To establish this we need only employ the well-known principles that (a) all tangents from a point to a sphere are equal in length, and (b) all tangents are equal that are drawn to a sphere from points equidistant from its centre. In both figures all points of the cone's bases are evidently equidistant from the centres of the tangent spheres.

140. On the upper nappe (Fig. 90) let $SH$ be the circle of contact of a sphere which is tangent at $F_1$ to the cutting plane $PLK$. The plane $PH_1$ of the circle cuts the plane of section in $Pm$. If $D$ is *any* point of the curve $DJE$, $J$ another point, and we can prove the ratio constant (and greater than unity) between the distances of $D$ and $J$ from $F_1$ and their distances to $Pm$, then the curve $DJE$ must be an *hyperbola*, by the Boscovich definition; $F_1$ must be the focus and $Pm$ the directrix.

$DF_1$ is a tangent whose real length is seen at $XS$. $JF_1$ and $JS$ are equal, being tangents to the sphere from the same point. We have then the proportion $XS:GR::JS:JR$, or $DF_1:DZ::JF_1:JR$. Since $JS$ and its equal $JF_1$ are greater than $JR$, and the ratio $JF_1$ to $JR$ is constant, the proposition is established.

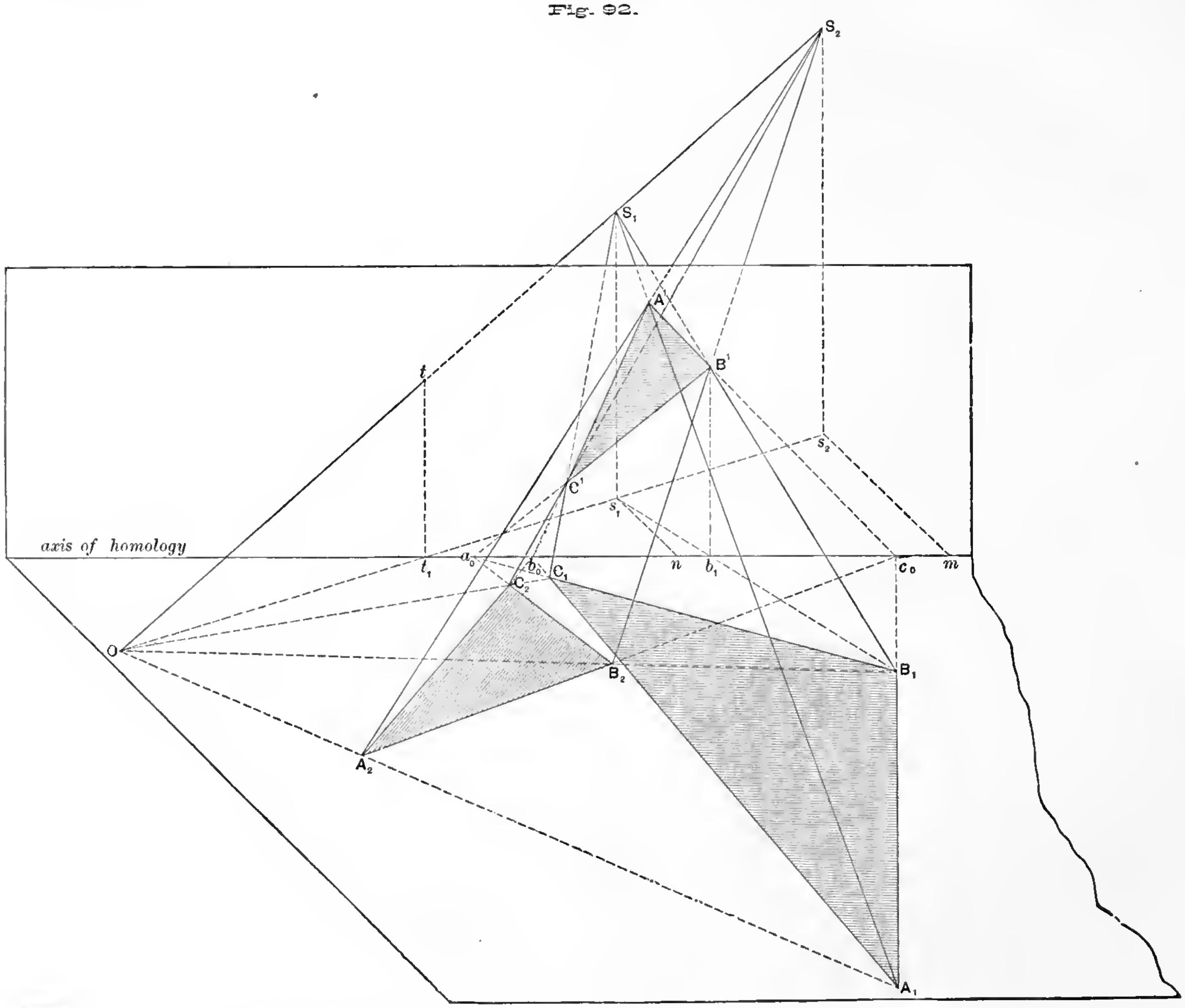
141. For the parabola on the lower nappe, since the plane $Mqk$ is inclined at the angle $\theta$ made by the elements, we have $QA=QB$ (opposite equal angles), and $QB$ equal $QF$ (equal tangents). $MF=BW=Mo$, therefore $MF:Mo::QF:QB(=QA)$, and it is as far from $M$ to the focus $F$ as to the directrix $sx$, fulfilling the condition essential for the parabola.

142. For the ellipse $KsH$, (Fig. 91), we have $PX$ and $NT$ as the lines to be proven directrices, and $F$ and $F_1$ the points of tangency of two spheres. Let $s$ be any point of the curve under consideration, and $VL$ the element containing $s$. This element cuts the contact circles of the spheres in $a$ and $A$. A plane through the cone's axis and parallel to the paper would contain $o\,t$, $o\,v$ and $v\,n$. Prolong $v\,n$ to meet a line $VR$ that is parallel to $KH$. Join $R$ with $a$, producing it to meet $PX$ at $r$. In the triangles $a\,s\,r$ and $a\,VR$ we have $s\,a:s\,r::Va:VR$. But $s\,a=s\,F$ (equal tangents) and similarly $Va=Vn$; hence $s\,F:s\,r::Vn:VR$, which ratio is less than unity; therefore $s$ is a point on an ellipse.*

The plane of the intersecting lines $Va$ and $Rr$ cuts the plane $MN$ in $A\,T$, which is therefore parallel to $a\,r$; hence $s\,A:s\,T::Vn:VR$. But $s\,A=s\,F_1$; therefore $s\,F_1:s\,T::Vn:VR$, the same ratio as before.

143. If the plane of section $PX$ were to approach parallelism to $VC$ the point $R$ would advance toward $n$, and when $VR$ became $Vn$ the plane would have reached the position to give the parabola.

Fig. 92.




*Schlömilch, *Geometrie des Maasses*, 1874.

144. The proof that $KsH$ (Fig. 91) is an ellipse when the curve is referred to *two foci* is as follows: $KF = Km$; $KF_1 = Kt$; therefore $KF + KF_1 = Km + Kt = tm = xn = 2KF + FF_1 = 2HF + FF_1$; i. e., $KF = HF_1$.

Since $sF = sa$ and $sF_1 = sA$ we have $sF + sF_1 = sa + sA = Aa = tm = xn = HK$. The sum of the distances from any point $s$ to the two fixed points $F$ and $F_1$ is therefore constant, and equal to the longer axis, $HK$.

## HOMOLOGOUS PLANE AND SPACE FIGURES.

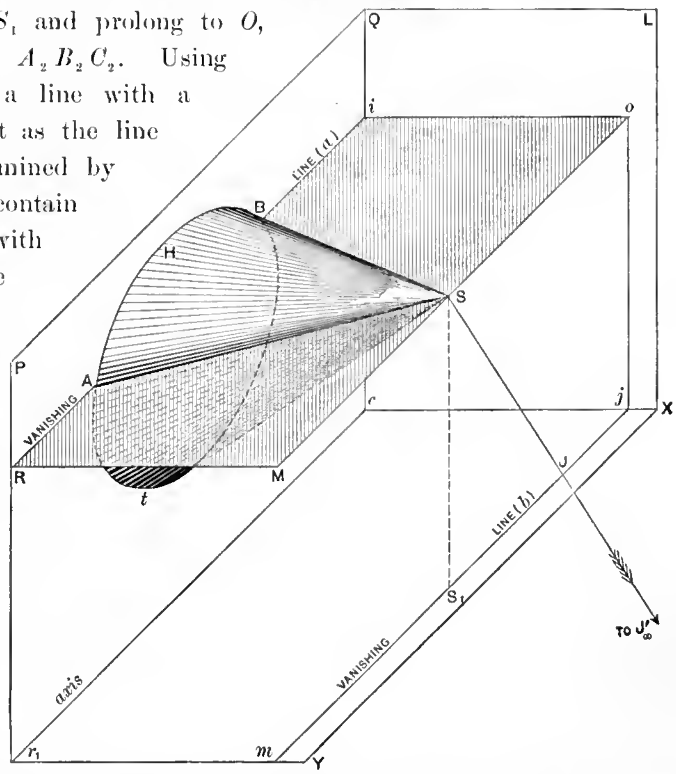
145. Before leaving the conic sections their construction will be given by the methods of Projective Geometry. (At this point review Arts. 4 and 9).

In Fig. 92, if $S_1$ is a centre of projection, then the figure $A^1B^1C^1$ is the *central projection* of $A_1B_1C_1$. The points $A^1$ and $A_1$ are *corresponding points*, being in different planes but collinear with $S_1$. Similarly $B^1$ corresponds to $B_1$, and $C^1$ to $C_1$.

With $S_2$ as the centre of projection we have the figure $A_2B_2C_2$ corresponding to $A^1B^1C^1$.

We are to show that some point $O$ can be found, *in the plane* of the figures $A_1B_1C_1$ and $A_2B_2C_2$, to which—and to each other—those figures bear the same relation as that existing between each of them and $A^1B^1C^1$ when considered in connection with one of the $S$-centres. This compels the points of intersection of corresponding *lines* to be collinear. Figures standing in these relations to a point and a line in their plane are called *homologous*. The *point* is called a *centre of homology;* the *line* an *axis of homology*. Points collinear with $O$, as $A_1$ and $A_2$, are *homologous points*.

Fig. 93.




To illustrate these statements join $S_2$ with $S_1$ and prolong to $O$, to meet the plane of the figures $A_1B_1C_1$ and $A_2B_2C_2$. Using the technical term *trace* for the intersection of a line with a plane or of one plane with another, we see that as the line $A_1A_2$ is the horizontal trace of the plane determined by the lines joining $A^1$ with $S_1$ and $S_2$ it must contain the horizontal trace, $O$, of the line joining $S_2$ with $S_1$. But this puts $A_2$ and $A_1$ into the same relation with $O$ that $A_2$ and $A^1$ sustain to $S_2$; or that of $A^1$ and $A_1$ to $S_1$.

Again, $A^1c_0$ is the trace, on the vertical plane, of any plane containing $A^1B^1$. This plane cuts the "axis of homology," $t_1m$, in $c_0$. As $A_1B_1$ lies in the plane of $S_1$ and $A^1B^1$, and in the horizontal plane as well, it can only meet the vertical plane in $c_0$, the point of intersection of all these planes. Similarly we find that $A^1C^1$ and $A_1C_1$, if prolonged, meet the axis at the point $b_0$; correspondingly $B^1C^1$ and $B_1C_1$ meet at $a_0$. But $A^1B^1$ and $A_2B_2$, being corresponding lines, lie in the plane with $S_2$, though belonging to figures in two other planes; they must, therefore, meet also at the same point, $c_0$; and similarly for the other lines in the figures used with $S_2$.

146. Were $A_1B_1C_1$ a *circle*, and all its points joined with $S_1$, the figure $A^1B^1C^1$ would obviously be an ellipse; equally so were $A_2B_2C_2$ a circle used in connection with $S_2$. We may, therefore,

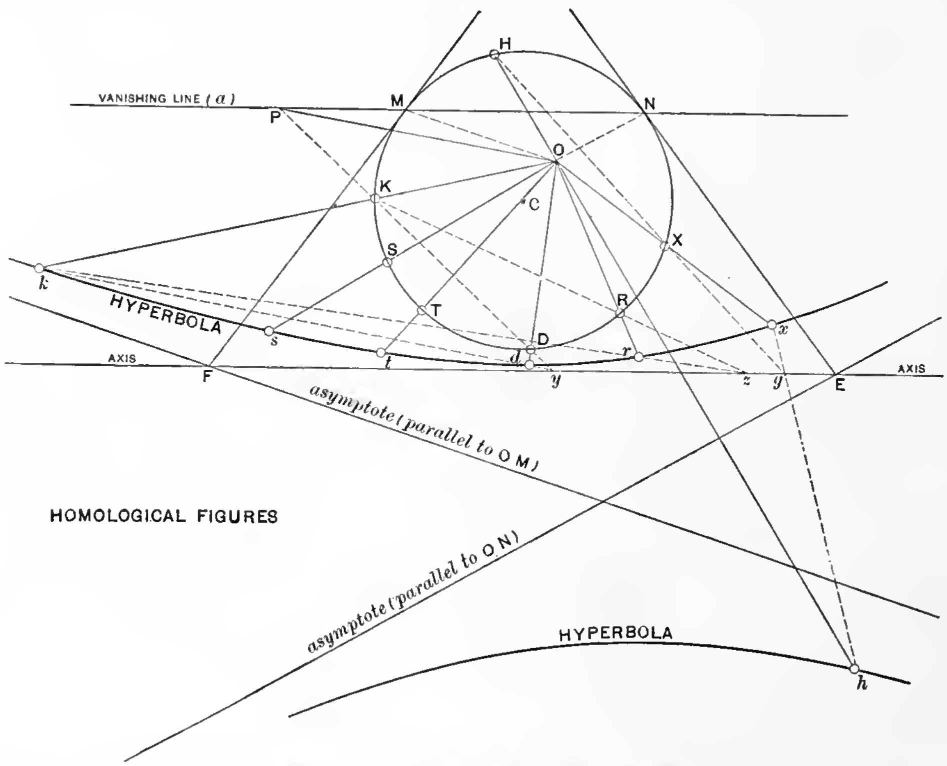
substitute a circle for $A_2 B_2 C_2$, and using $O$ on the same plane with it get an ellipse in place of the triangle $A_1 B_1 C_1$. Before illustrating this it is necessary to show the relation of the axis to the other elements of the problem, and supply a test as to the nature of the conic.

147. First as to the *axis*, and employing again for a time a space figure (Fig. 93), it is evident that raising or lowering the horizontal plane $cXY$ parallel to itself, and with it, necessarily, the axis, would not alter the *kind* of curve that it would cut from the cone $S.HAB$, were the elements of the latter prolonged. But raising or lowering the *centre* $S$, while the base circle $AHBt$ remained, as before, in the same place, would decidedly affect the curve. Where it is, there are two elements of the cone, $SA$ and $SB$, which would never meet the plane $cXY$. The shaded plane containing those elements meets the vertical plane in "vanishing line (a)," parallel to the axis. This contains the projections, $A$ and $B$, of the points at infinity where the lower plane may be considered as cutting the elements $SA$ and $SB$. Were $S$ and the shaded plane raised to the level of $H$, making "vanishing line (a)" tangent to the base, there would be one element, $SH$, of the cone, parallel to the lower plane, and the section of the cone by the latter would be the *parabola;* as it is, the *hyperbola* is indicated. The former would have but one point at infinity; the latter, two.

148. Raise the centre $S$ so that the vanishing line does not cut the base, and evidently no line from $S$ to the base would be parallel to the lower plane; but the latter would cut all the elements on one side of the vertex, giving the ellipse.

149. Bearing in mind that the projection of the circle $AHBt$ is on the lower plane produced, if we wish to bring both these figures and the centre $S$ into one plane without destroying the relation between them, we may imagine the end plane $QLX$ removed, the rotation of the remaining system

Fig. 94.




occurring about $cr_1$ in a manner exactly similar to that which would occur were $iojc$ a system of four pivoted links, and the point $o$ pressed toward $c$. The motion of $S$ would be parallel and equal

to that of $o$, and, like the latter, $S$ would evidently maintain its distance from the vanishing line and describe a circular arc about it. The vanishing line would remain parallel to the axis.

150. From the foregoing we see that to obtain the hyperbola, by projection of a circle from a point in the plane of the latter, we would require simply a *secant* vanishing line, $MN$ (Fig. 94), and an axis of homology parallel to it. Take any point $P$ on the vanishing line and join it with any point $K$ of the circle. $PK$ meets the axis at $y$; hence whatever line *corresponds* to $PK$ must also meet the axis at $y$. $OP$ is analogous to $SA$ of Fig. 93, in that it meets its corresponding line at infinity, i.e., is parallel to it. Therefore $yk$, parallel to $OP$, corresponds to $Py$, and meets the ray $OK$ at $k$, corresponding to $K$. Then $K$ joined with any other point $R$ gives $Kz$. Join $z$ with $k$ and prolong $OR$ to intersect $kz$, obtaining $r$, another point of the hyperbola.

151. In Fig. 93, were a tangent drawn to arc $AHB$ at $B$ it would meet the axis in a point which, like all points *on* the axis, "corresponds to itself." From that point the projection of that tangent on the lower plane would be parallel to $SB$, since they are to meet at infinity. Or, if $SJ$ is parallel to the tangent at $B$, then $J$ will be the projection of $J^1$ at infinity, where $SJ$ meets the tangent; $J$ will be therefore one point of the projection of said tangent on the lower plane, while another point would be, as previously stated, that in which the tangent at $B$ meets the axis.

152. Analogously in Fig. 94, the tangents at $M$ and $N$ meet the axis, as at $F$ and $E$; but the projectors $OM$ and $ON$ go to points of tangency *at infinity*; $M$ and $N$ are on a "vanishing line"; hence $OM$ is parallel to the tangent at infinity, that is, to the *asymptote* (see Art. 134) through $F$; while the other asymptote is a parallel through $E$ to $ON$.

153. As in Fig. 93 the projectors from $S$ to all points of the arc above the level of $S$ could cut the lower plane only by being produced to the right, giving the right-hand branch of the hyperbola; so, in Fig. 94, the arc $MHN$, above the vanishing line, gives the lower branch of the hyperbola. To get a point of the latter, as $h$, and having already obtained any point $x$ of the other branch, join $H$ with $X$ (the original of $x$) and get its intersection, $g$, with the axis. Then $xgh$ corresponds to $gXH$, and the ray $OH$ meets it at $h$, the projection of $H$.

The cases should be worked out in which the vanishing line is tangent to the circle or exterior to it.

154. The homological figures with which we have been dealing were *plane* figures. But it is possible to have *space* figures homological with each other.

In homological space figures corresponding lines meet at the same point *in a plane*, instead of the same point on a line. A vanishing *plane* takes the place of a vanishing *line*. The figure that is in homology with the original figure is called the *relief-perspective* of the latter. (See Art. 11.)

Remarkably beautiful effects can be obtained by the construction of homological space figures, as a glance at Fig. 95 will show. The figure represents a triple row of groined arches, and is from a photograph of a model designed by Prof. L. Burmester.

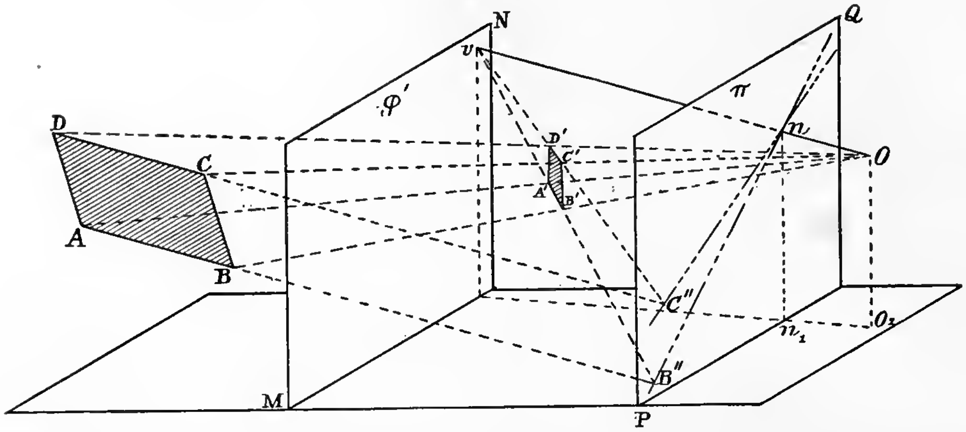
Fig. 95.

Although not always requiring the use of the irregular curve and therefore not strictly the material for a topic in this chapter, its close analogy to the foregoing matter may justify a few words at this point on the construction of a relief-perspective.

155. In Fig. 96 the plane $PQ$ is called the *plane of homology* or *picture-plane*, and—adopting Cremona's notation—we will denote it by $\pi$. The *vanishing plane* $MN$, or $\phi'$, is parallel to it. $O$ is

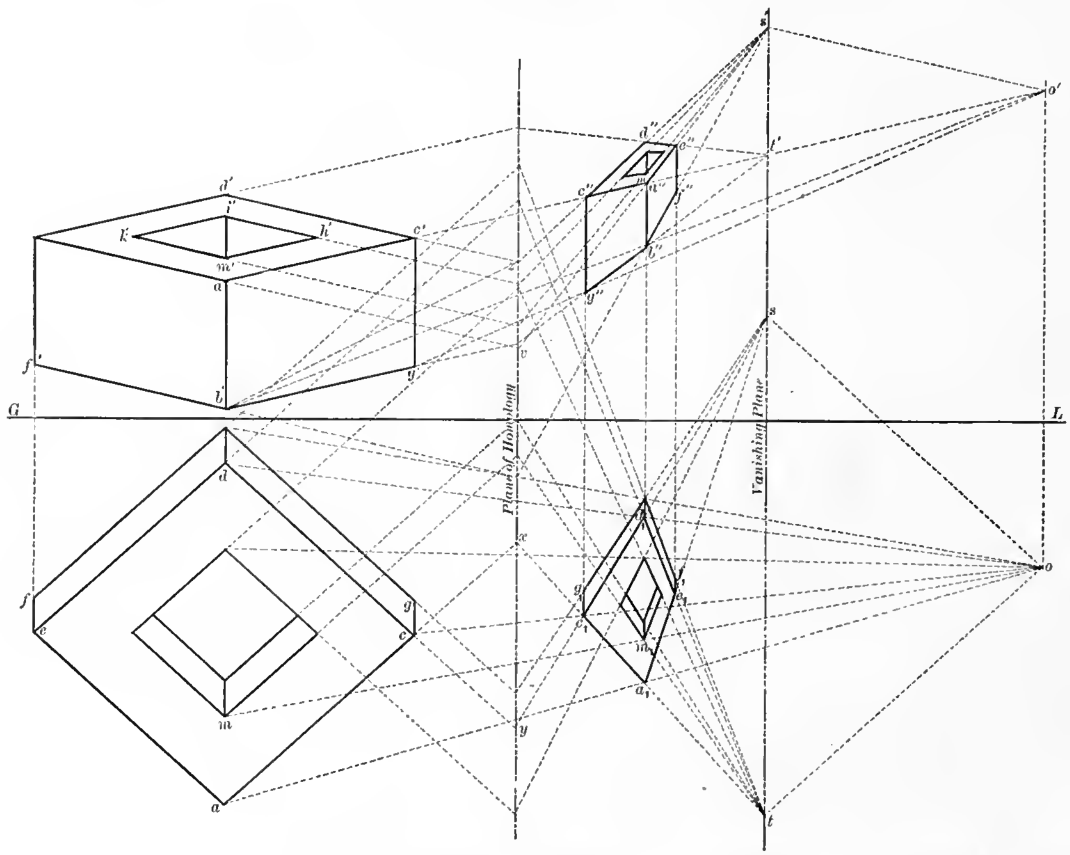
the *centre of homology* or *perspective-centre*. All points in the plane $\pi$ are their own perspectives, or, in other words, *correspond to themselves*. Therefore $B''$ is one point of the projection or perspective of the line $A\,B$, being the intersection of $A\,B$ with $\pi$. The line $O\,v$, parallel to $A\,B$, would meet the

Fig. 96.




latter *at infinity*; therefore $v$, in the vanishing plane $\phi'$, would be the projection upon it of the point at infinity. Joining $v$ with $B''$, and cutting $v\,B''$ by rays $O\,A$ and $O\,B$, gives $A'\,B'$ as the relief-perspective of $A\,B$. The plane through $O$ and $A\,B$ cuts $\pi$ in $B''\,n$, which is an *axis of homology* for $A\,B$ and $A'\,B'$, exactly as $m\,n$ in Fig. 92 is for $A_1\,B_1$ and $A_2\,B_2$.

Fig. 97.




As $D\,C$ in Fig. 96 is parallel to $A\,B$, a parallel to it through $O$ is again the line $O\,v$.

The trace of $DC$ on $\pi$ is $C''$. Joining $v$ with $C''$ and cutting $v\,C''$ by rays $O\,D$, $O\,C$, obtains $D'\,C'$ in the same manner as $A'\,B'$ was derived. The originals of $A'\,B'$ and $C'\,D'$ are parallel lines; but we see that their relief-perspectives meet at $v$. The vanishing plane is therefore the locus* of the vanishing points of lines that are parallel on the original object, while the plane of homology is the locus of the axes of homology of corresponding lines; or, differently stated, any line and its relief-perspective will, if produced, meet on the plane of homology.

156. Fig. 97 is inserted here for the sake of completeness, although its study may be reserved, if necessary, until the chapter on projections has been read. In it a solid object is represented at the left, in the usual views, plan and elevation; $G\,L$ being the *ground line* or axis of intersection of the planes on which the views are made. The planes $\pi$ and $\phi'$ are interchanged, as compared with their positions in Fig. 96, and they are seen as lines, being assumed as perpendicular to the paper. The relief-perspective appears between them, in plan and elevation.

The lettering of $A\,B$ and $D\,C$, and the lines employed in getting their relief-perspectives, being identical with the same constructions in Fig. 96, ought to make the matter clear at a glance to all who have mastered what has preceded.

Burmester's *Grundzüge der Relief-Perspective* and Wiener's *Darstellende Geometrie* are valuable reference works on this topic for those wishing to pursue its study further; but for special work in the line of homological *plane* figures the student is recommended to read Cremona's *Projective Geometry* and Graham's *Geometry of Position*, the latter of which is especially valuable to the engineer or architect, since it illustrates more fully the practical application of central projection to Graphical Statics.

## LINK-MOTION CURVES

157. *Kinematics* is the science which treats of *pure motion, regardless of the cause or the results of the motion.*

It is a purely kinematic problem if we lay out on the drawing-board the path of a point on the connecting-rod of a locomotive, or of a point on the piston of an oscillating cylinder, or of any point on one of the moving pieces of a mechanism. Such problems often arise in machine design, especially in the invention or modification of valve-motions.

Some of the motion-curves or point-paths that are discovered by a study of relative motion are without special name. Others, whose mathematical properties had already been investigated and the curves dignified with names, it was later found could be mechanically traced. Among these the most familiar examples are the Ellipse and the Lemniscate, the latter of which is employed here to illustrate the general problem.

The moving pieces in a mechanism are rigid and inextensible, and are always under certain *conditions of restraint.* "Conditions of restraint" may be illustrated by the familiar case of the connecting-rod of the locomotive, one end of which is always attached to the driving-wheel at the crank-pin and is therefore constrained to describe a circle about the axle of that wheel, while the other end of the rod must move in a straight line, being fastened by the "wrist-pin" to the "cross-head," which slides between straight "guides." The first step in tracing a point-path of any mechanism is therefore the determination of the *fixed* points, and a general analysis of the motion.

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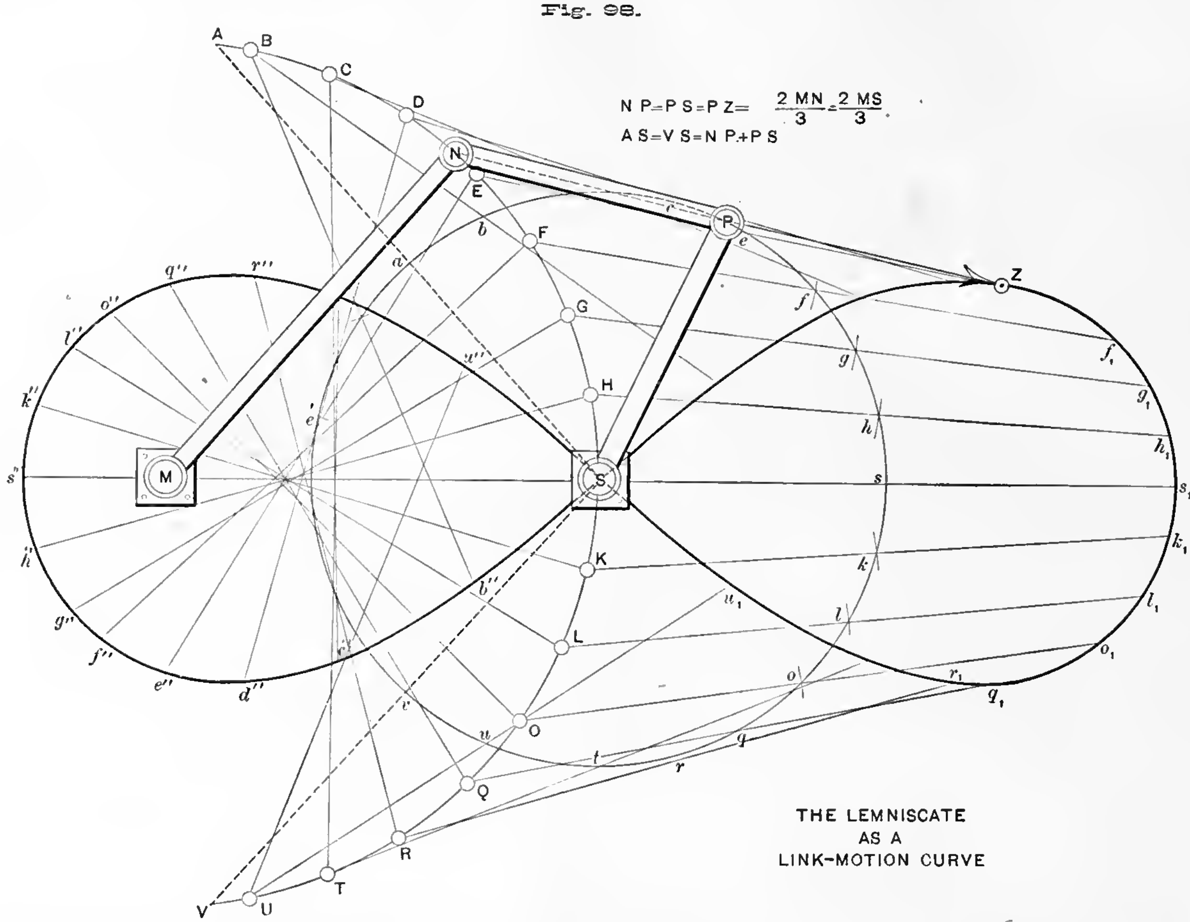
* *Locus* is the Latin for *place;* and in rather untechnical language, although in the exact sense in which it is used mathematically, we may say that the *locus* of points or lines is the place where you may expect to find them under their conditions of restriction. For example, the surface of a sphere is the *locus* of all points equidistant from a fixed point (its centre). The *locus* of a point moving in a plane so as to remain at a constant distance from a given fixed point, is a circle having the latter point as its centre.

158. We have given, in Fig. 98, two links or bars, $MN$ and $SP$, fastened at $N$ and $P$ by pivots to a third link, $NP$, while their other extremities are pivoted on *stationary* axes at $M$ and $S$. The only movement possible to the point $N$ is therefore in a circle about $M$; while $P$ is equally limited to circular motion about $S$. The points on the link $NP$, with the exception of its

Fig. 98.




extremities, have a compound motion, in curves whose form it is not easy to predict and which differ most curiously from each other. The figure-of-eight curve shown, otherwise the "Lemniscate of Bernoulli," is the point-path of $Z$, the link $NP$ being supposed prolonged by an amount, $PZ$, equal to $NP$. Since $NP$ is constant in length, if $N$ were moved along to $F$, the point P would have to be at a distance $NP$ from $F$ and also on the circle to which it is confined; therefore its new position $f$, is at the intersection of the circle $Psr$ by an arc of radius $PN$, centre $F$. Then $Ff$ prolonged by an amount equal to itself, gives $f_1$, another point of the Lemniscate, and to which $Z$ has then moved. All other positions are similarly found.

If the motion of $N$ is toward $D$ it will soon reach a limit, $A$, to its further movement in that direction, arriving there at the instant that $P$ reaches $a$, when $NP$ and $PS$ will be in one straight line, $SA$. In this position any movement of $P$ either side of $a$ will drag $N$ back over its former path; and unless $P$ moves to the left, past $a$, it would also retrace its path. $P$ reaches a similar "dead point" at $v$.

To obtain a Lemniscate the links $NP$ and $PS$ had to be equal, as also the distance $MS$ to $MN$. By varying the proportions of the links, the point-paths would be correspondingly affected.

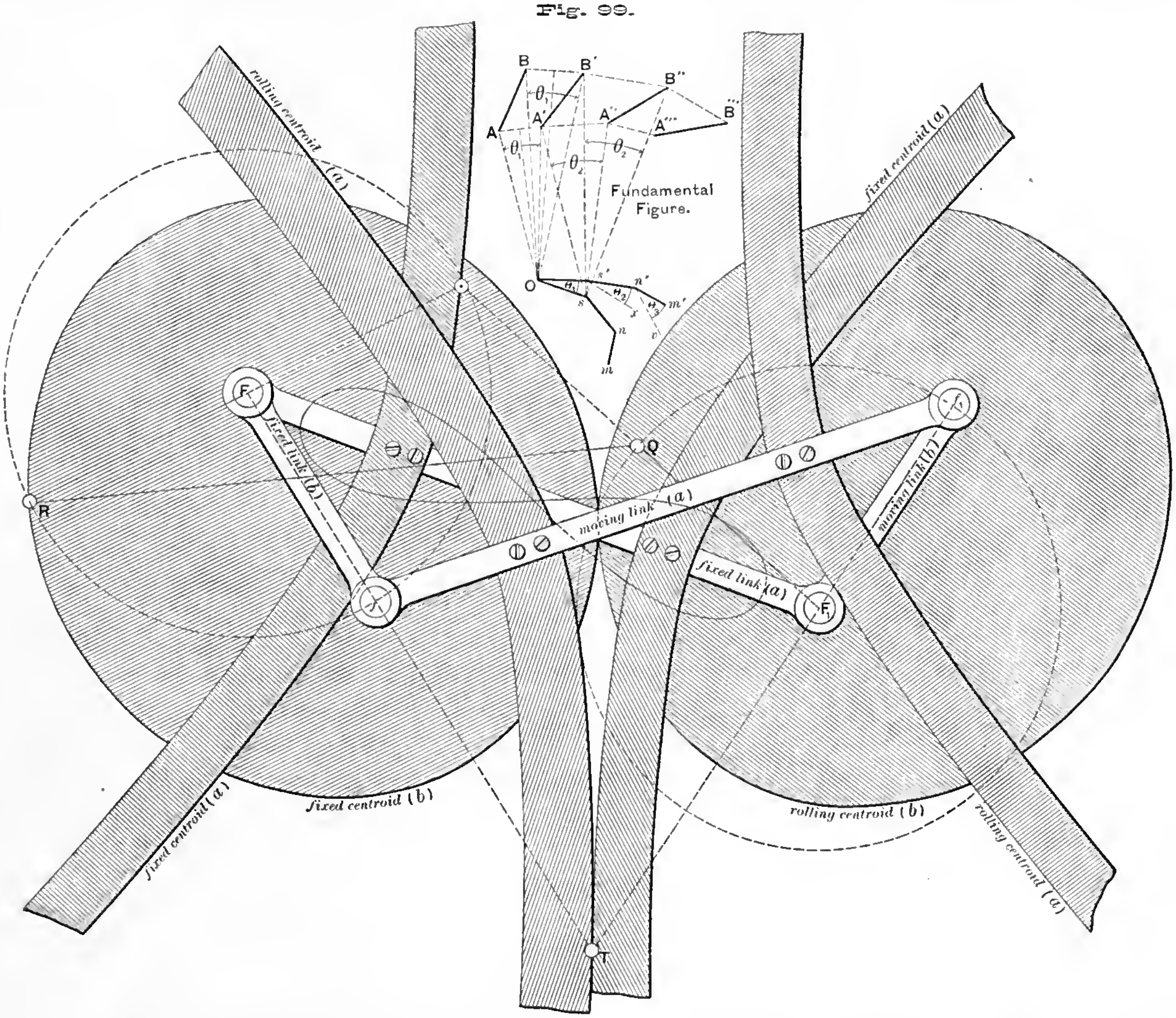
By tracing the path of a point on $PN$ produced, and as far from $N$ as $Z$ is from $P$, the student will obtain an interesting contrast to the Lemniscate.

If $M$ and $S$ were joined by a link, and the latter held rigidly in position, it would have been called the *fixed link;* and although its use would not have altered the motions illustrated, and it is not essential that it should be drawn, yet in considering a mechanism as a whole, the line joining the fixed centres always exists, in the imagination, as a link of the complete system.

### INSTANTANEOUS CENTRES.—CENTROIDS.

159. Let us imagine a boy about to hurl a stone from a sling. Just before he releases it he runs forward a few steps, as if to add a little extra impetus to the stone. While taking those few steps a peculiar shadow is cast on the road by the end of the sling, if the day is bright. The

Fig. 99.




boy moves with respect to the earth; his hand moves in relation to himself, and the end of the sling describes a *circle* about his hand. The last is the only definite element of the three, yet it is sufficient to simplify otherwise difficult constructions relating to the complex curve which is described relatively to the earth.

A tangent and a normal to a *circle* are easily obtained, the former being, as need hardly be stated at this point, perpendicular to the radius at the point of tangency, while the normal simply coincides in direction with such radius. If the stone were released at any instant it would fly off in a straight line, tangent to the circle it was describing about the hand as a centre; but such line would, at the instant of release, be tangent also to the compound curve. If, then, we wish a tangent at a given point of any curve generated by a point in motion, we have but to reduce that motion to circular motion about some moving centre; then, joining the point of desired tangency with the—at that instant—position of the moving centre, we have the *normal*, a perpendicular to which gives the tangent desired.

A centre which is thus used *for an instant only* is called an *instantaneous centre*.

160. In Fig. 99 a series of instantaneous centres are shown and an important as well as interesting fact illustrated, viz., that every moving piece in a mechanism might be rigidly attached to a certain curve, and by the rolling of the latter upon another curve the link might be brought into all the positions which its visible modes of restraint compel it to take.

161. In the "Fundamental" part of Fig. 99 $AB$ is assumed to be one position of a link. We next find it, let us suppose, at $A'B'$, $A$ having moved over $AA'$, and $B$ over $BB'$. Bisecting $AA'$ and $BB'$ by perpendiculars intersecting at $O$, and drawing $OA$, $OA'$, $OB$ and $OB'$, we have $AOA' = \theta_1 = BOB'$, and $O$ evidently a point about which, as a centre, the turning of $AB$ through the angle $\theta_1$ would have brought it to $A'B'$. Similarly, if the next position in which we find $AB$ is $A''B''$, we may find a point $s$ as the centre about which it might have turned to bring it there; the angle being $\theta_2$, probably different from $\theta_1$. $N$ and $m$ are analogous to $O$ and $s$.

If $Os'$ be drawn equal to $Os$ and making with the latter an angle $\theta_1$, equal to the angle $AOA'$, and if $Os$ were rigidly attached to $AB$, the latter would be brought over to $A'B'$ by bringing $Os'$ into coincidence with $Os$. In the same manner, if we bring $s'n'$ upon $sn$ through an angle $\theta_2$ about $s$, then the next position, $A''B''$, would be reached by $AB$. $O's'n'm'$ is then part of a polygon whose rolling upon $Osnm$ would bring $AB$ into all the positions shown, provided the polygon and the line were so attached as to move as one piece. Polygons whose vertices are thus obtained are called *central polygons*.

If *consecutive* centres were joined we would have curves, called *centroids**, instead of polygons; the one corresponding to $Osnm$ being called the *fixed*, the other the *rolling* centroid. The perpendicular from $O$ upon $AA'$ is a normal to that path. But were $A$ to move in a *circle*, the normal to its path at any instant would be simply the radius to the position of $A$ at that instant.

If, then, both $A$ and $B$ were moving in *circular* paths, we would find the instantaneous centre at the intersection of the normals (radii) at the points $A$ and $B$.

162. In Fig. 98 the instantaneous centre about which the whole link $NP$ is turning is at the intersection of radii $MN$ and $SP$ (produced); and calling it $X$ we would have $XZ$ for the normal at $Z$ to the Lemniscate.

163. The shaded portions of Fig. 99 illustrate some of the forms of centroids.

The mechanism is of four links, opposite links equal. Unlike the usual quadrilateral fulfilling this condition, the long sides cross, hence the name "anti-parallelogram."

The "fixed link (a)" corresponds to $MS$ of Fig. 98, and its extremities are the centres of rotation of the short links, whose ends, $f$ and $f_1$, describe the dotted circles.

For the given position $T$ is evidently the instantaneous centre. Were a bar pivoted at $T$ and

*Reuleaux' nomenclature; also called *centrodes* by a number of writers on Kinematics.

fastened at right angles to "moving link (a)," an *infinitesimal* turning about $T$ would move "link (a)" exactly as under the old conditions.

By taking "link (a)" in all possible positions, and, for each, prolonging the radii through its extremities, the points of the fixed centroid are determined. Inverting the combination so that "moving link (a)" and its opposite are interchanged, and proceeding as before, gives the points of "rolling centroid (a)."

These centroids are branches of hyperbolas having the extremities of the long links as foci.

By holding a short link stationary, as "fixed link (b)," an elliptical fixed centroid results; "rolling centroid (b)" being obtained, as before, by inversion. The foci are again the extremities of the fixed and moving links.

Obviously the curved pieces represented as screwed to the links would not be employed in a practical construction, and they are only introduced to give a more realistic effect to the figure and possibly thereby conduce to a clearer understanding of the subject.

164. It is interesting to notice that the Lemniscate occurs here under new conditions, being traced by the middle point of "moving link (a)."

The study of kinematics is both fascinating and profitable, and it is hoped that this brief glance at the subject may create a desire on the part of the student to pursue it further in such works as Reauleaux' *Kinematics of Machinery* and Burmester's *Lehrbuch der Kinematik.*

165. Before leaving this topic the important fact should be stated, which now needs no argument to establish, that the instantaneous centre, for any position of a moving piece, is the *point of contact* of the rolling and fixed centroids. We shall have occasion to use this principle in drawing tangents and normals to the

TROCHOIDS

which are the principal *Roulettes*, or *roll-traced curves*, and which may be defined as follows:—

If, in the same plane, one of two circles roll upon the other without sliding, the path of any point on a radius of the rolling circle or on the radius produced is a *trochoid.*

166. *The Cycloid.* Since a straight line may be considered a circle of *infinite radius*, the above definition would include the curve traced by a point on the circumference of a locomotive wheel as it rolls along the rail, or of a carriage wheel on the road. This curve is known as a *cycloid** and is shown in $T\,n\,a\,b\,c$, Fig. 100. It is the proper outline for a portion of each tooth in a certain case of gearing, viz., where one wheel has an infinite radius, that is, becomes a "rack."

Were $T_6$ a ceiling-corner of a room, and $T_{12}$ the diagonally opposite floor-corner, a weight would slide from $T_6$ to $T_{12}$ more quickly on guides curved in cycloidal shape than if shaped to any other curve, or if straight. If started at $c$, or any other point of the curve, it would reach $T_{12}$ as soon as if started at $T_6$.

167. In beginning the construction of the cycloid we notice, first, that as $T\,V\,D$ rolls on the straight line $A\,B$, the arrow $D\,R\,T$ will be reversed in position (as at $D_5\,T_6$) as soon as the semi-circumference $T\,3\,D$ has had rolling contact with $A\,B$. The *tracing point* will then be at $T_6$, its maximum distance from $A\,B$.

When the wheel has rolled itself out once upon the rail, the point $T$ will again come in contact with the rail, as at $T_{12}$.

*"Although the invention of the cycloid is attributed to Galileo, it is certain that the family of curves to which it belongs had been known and some of the properties of such curves investigated, nearly two thousand years before Galileo's time, if not earlier. For ancient astronomers explained the motion of the planets by supposing that each planet travels uniformly round a circle whose centre travels uniformly around another circle."—Proctor, *Geometry of Cycloids.*

The distance $T T_{12}$ evidently equals $2 \pi r$, when $r = T R$. We also have $T D_5 = D_5 T_{12} = \pi r$.

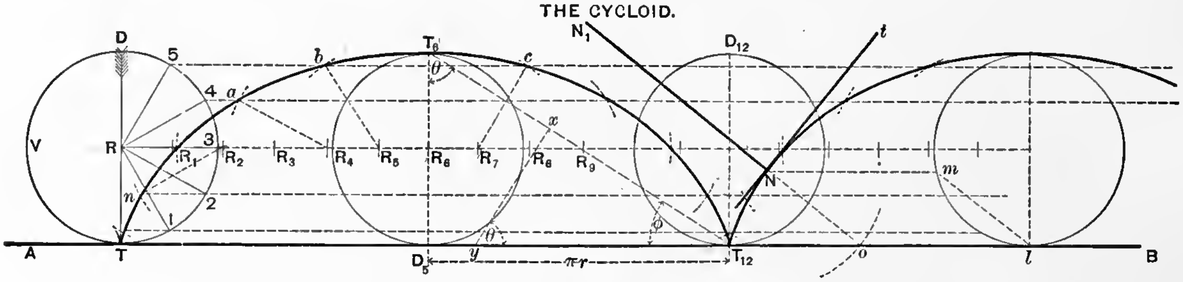
If the semi-circumference $T 3 D$ (equal to $\pi r$) be divided into any number of equal parts, and also the path of centres $R R_6$ (again $= \pi r$) into the same number of equal parts, then as the points 1, 2, etc., come in contact with the rail, the centre $R$ will take the positions $R_1, R_2$, etc., directly above the corresponding points of contact. A sufficient rolling of the wheel to bring point 2 upon $A B$ would evidently raise $T$ from its original position to the former level of 2. But as $T$ must always be at a radius' distance from $R$, and the latter would by that time be at $R_2$, we would find $T$ located at the intersection ($n$) of the dotted *line of level* through 2 by an arc of radius $R T$, centre $R_2$. Similarly for other points.

The construction, summarized, involves the drawing of *lines of level* through *equidistant* points of division on a semi-circumference of the rolling circle, and their intersection by arcs of constant radius (that of the rolling circle) from centres which are the successive positions taken by the centre of the rolling circle.

It is worth while calling attention to a point occasionally overlooked by the novice, although almost self-evident, that, in the position illustrated in the figure, the point $T$ drags behind the centre $R$ until the latter reaches $R_6$, when it passes and goes ahead of it. From $R_7$ the line of level through 5 could be cut not alone at $c$ by an arc of radius $c R_7$ but also in a second point; evidently but one of these points belongs to the cycloid, and the choice depends upon the direction of turning, and upon the relative position of the rolling centre and the moving point. This matter requires more thought in drawing trochoidal curves in which both circles have finite radii, as will appear later.

Fig. 100.

THE CYCLOID.




168. Were points $T_6$ and $T_{12}$ given, and the semi-cycloid $T_6 T_{12}$ desired, we can readily ascertain the "base," $A B$, and generating circle, as follows: Join $T_6$ with $T_{12}$; at any point of such line, as $x$, erect a perpendicular, $x y$; from the similar triangles $x y T_{12}$ and $T_6 D_5 T_{12}$, having angle $\phi$ common and angles $\theta$ equal, we see that

$$x y : x T_{12} :: T_6 D_5 : D_5 T_{12} :: 2 r : \pi r :: 2 : \pi :: 1 : \frac{\pi}{2}; \text{ or, very nearly, as } 14:22.$$

If, then, we lay off $x T_{12}$ equal to *twenty-two* equal parts on any scale, and a perpendicular, $x y$, *fourteen* parts of the same scale, the line $y T_{12}$ will be the base of the desired curve; while the diameter of the generating circle will be the perpendicular from $T_6$ to $y T_{12}$ prolonged.

169. To draw the tangent to a cycloid at any point is a simple matter, if we see the analogy between the *point of contact* of the wheel and rail at any instant, and the *hand* used in the former illustration (Art. 159). At any one moment each point on the entire wheel may be considered as describing an infinitesimal arc of a circle whose radius is the line joining the point with the point of contact on the rail. The tangent at $N$, for example, (Fig. 100), would be $t N$, perpendicular to the normal, $N o$, joining $N$ with $o$; the latter point being found by using $N$ as a centre and

cutting $AB$ by an arc of radius equal to $ml$, in which $m$ is a point *at the level* of $N$ on *any* position of the rolling circle, while $l$ is the corresponding point of contact. The point $o$ might also have been located by the following method: Cut the line of centres by an arc, centre $N$, radius $TR$; $o$ would obviously be vertically below the position of the rolling centre thus determined.

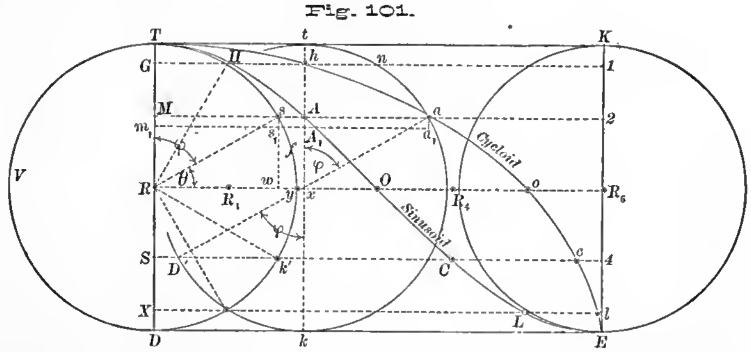
170. *The Companion to the Cycloid.* The kinematic method of drawing tangents, just applied, was devised by Roberval, as also the curve named by him the "Companion to the Cycloid," to which allusion has already been made (Art. 120) and which was invented by him in 1634 for the purpose of solving a problem upon which he had spent six years without success, and which had foiled Galileo, viz., the calculating of the area between a cycloid and its base. Galileo was reduced to the expedient of comparing the area of the cycloid with that of the rolling circle by weighing paper models of the two figures. He concluded that the area in question was nearly but not exactly three times that of the rolling circle. That the latter would have been the correct solution may be readily shown by means of the "Companion," as will be found demonstrated in Art. 172.

171. Suppose two points coincident at $T$ (Fig. 101) and starting simultaneously to generate curves, the first of these points to trace the cycloid during the rolling of circle $TVD$, while the second is to move independently of the circle and so as to be always *at the level* of the point tracing the cycloid, yet at the same time *vertically above* the point of contact of the circle and base. This makes the second point always as far from the initial vertical diameter, or *axis*, of the cycloid, as the length of the arc from $T$ to whatever level the tracing point of the latter has then reached; that is, $MA$ equals arc $THs$; $RO$ equals quadrant $Tsy$.

Adopting the method of Analytical Geometry, and using $O$ as the *origin*, we may reach any point, $A$, on the curve, by *co-ordinates*, as $Ox$, $xA$, of which the horizontal is called an *abscissa*, the vertical an *ordinate*. By the preceding construction $Ox$ equals *arc* $sfy$, while $xA$ equals $sw$—the *sine* of the same arc. The "Companion" is therefore a *curve of sines* or *sinusoid*, since, starting from $O$, the abscissas are equal to or proportional to the arc of a circle, while the ordinates are the sines of those arcs. It is also the orthographic projection of a 45°-helix.

This curve is particularly interesting as "expressing the law of the vibration of perfectly elastic solids; of the vibratory movement of a particle acted upon by a force which varies directly as the distance from the origin; approximately, the vibratory movement of a pendulum; and exactly the law of vibration of the so-called mathematical pendulum."* (See also Art. 356).

Fig. 101.




172. From the symmetry of the sinusoid with respect to $RR_6$ and to $O$, we have *area* $TAOR = ECOR_6$; adding *area* $DELOR$ to both members we have the area between the sinusoid and $TD$ and $DE$ equal to the rectangle $RE$, or one-half the rectangle $DEKT$; or to $\frac{1}{2}\pi r \times 2r = \pi r^2$, the area of the rolling circle.

As $TACE$ is but half of the entire sinusoid it is evident that the total area below the curve is twice that of the generating circle.

The area between the cycloid and its "companion" remains to be determined, but is readily ascertained by noting that as any point of the latter, as $A$, is on the vertical diameter of the circle

* Wood, *Elements of Co-ordinate Geometry*, p. 209.

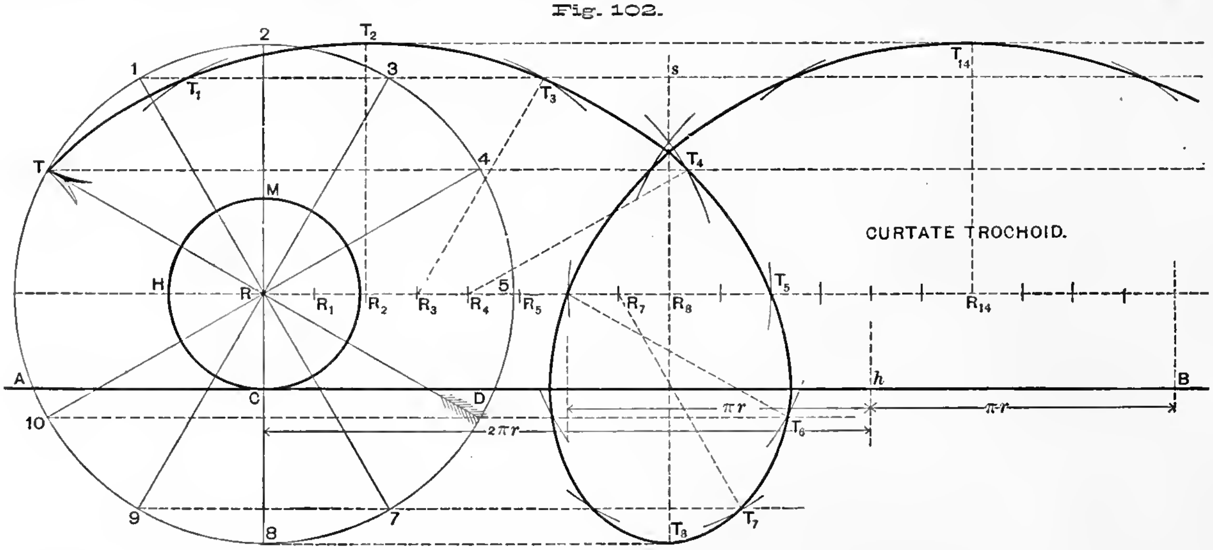
passing through the then position of the tracing point, as $a$, the distance, $A\,a$, between the two curves at any level, is merely the semi-chord of the rolling circle at that level. But this, evidently, equals $M\,s$, the semi-chord at the same level on the equal circle. The equality of $M\,s$ and $A\,a$ makes the elementary rectangles $M\,s\,s_1\,m_1$ and $A\,A_1\,a_1\,a$ equal; and considering all the possible similarly-constructed rectangles of infinitesimal altitude, the sum of those on semi-chords of the rolling circle would equal the area of the semi-circle $T\,D\,y$, which is therefore the extent of the area between the two curves under consideration.

The figure showing but half of a cycloid, the total area between it and its "companion" must be that of the rolling circle. Adding this to the area between the "companion" and the base makes the total area between cycloid and base equal to *three* times that of the rolling circle.

173. The paths of points carried by and in the plane of the rolling circle, though not on its circumference, are obtained in a manner closely analogous to that employed for the cycloid.

In Fig. 102 the looped curve, traced by the arrow-point while the circle $C\,H\,M$ rolls on the base $A\,B$, is called the *Curtate Trochoid.* To obtain the various positions of the tracing point $T$ describe a circle through it from centre $R$. On this circle lay off any even number of equal arcs, and draw radii from $R$ to the points of division; also "lines of level" through the latter. The radii drawn intercept equal arcs on the rolling circle $C\,H\,M$, whose straight equivalents are next laid off on the path of centres, giving $R_1$, $R_2$, etc. While the first of these arcs rolls upon $A\,B$ the point $T$ turns through the angle $T\,R\,1$ about $R$, and reaches the line of level through point 1. But $T$ is always at the distance $R\,T$ (called the *tracing radius*) from $R$; and, as $R$ has reached $R_1$ in the rolling supposed, we will find $T_1$—the new position of $T$—by an arc from $R_1$, radius $T\,R$, cutting said line of level.

Fig. 102.




After what has preceded, the figure may be assumed to be self-interpreting, each position of $T$ having been joined with the position of $R$ which determined it.

174. Were a tangent wanted at any point, as $T_7$, we have, as before, to determine the point of contact of rolling circle and line when $T$ reached $T_7$, and use it as an instantaneous centre. $T_7$ was obtained from $R_7$; and the point of contact must have been vertically below the latter and on $A\,B$. Joining such point to $T_7$ gives the normal, from which the tangent follows in the usual way.

175. *The Prolate Trochoid.* Had we taken a point *inside* of the circle $C\,H\,M$ and constructed its path the only difference between it and the curve illustrated would have been in the *name* and the

*shape* of the curve. An undulating, wavy path would have resulted, called the *prolate trochoid;* but, as before, we would have described a circle through the tracing point; divided it into equal parts; drawn lines of level, and cut them by arcs of constant radius, using as centres the successive positions of $R$. A bicycle pedal describes a prolate trochoid relatively to the earth.

### HYPO-, EPI- AND PERI-TROCHOIDS.

176. Circles of *finite* radius can evidently be tangent in but two ways—either *externally*, or *internally;* if the latter, the larger may roll on the one within it, or the smaller may roll inside the larger. When a small circle rolls within a larger the *radius* of the latter may be greater than the *diameter* of the rolling circle, or may equal it, or be smaller. On account of an interesting property of the curves traced by points in the planes of such rolling circles, viz., their capability of being generated, trochoidally, in two ways, a nomenclature was necessary which would indicate how each curve was obtained. This is included in the tabular arrangement of names below and which was the outcome of an investigation* made by the writer in 1887 and presented before the American Association for the Advancement of Science. In accepting the new terms, advanced at that time, Prof. Francis Reuleaux suggested the names *Ortho-cycloids* and *Cyclo-orthoids* for the classes of curves of which the cycloid and involute are respectively representative; *orthoids* being the paths of points in a fixed position with respect to a straight line rolling upon *any* curve, and *cyclo-orthoid* therefore implying a circular director or base-curve. These appropriate terms have been incorporated in the table.

For the last column a point is considered as *within* the rolling circle of infinite radius when on the normal to its initial position and on the side toward the centre of the fixed circle.

As will be seen by reference to the Appendix, the curves whose names are preceded by the same letter may be identical. Hence the terms *curtate* and *prolate*, while indicating whether the tracing point is beyond or within the circumference of the rolling circle, give no hint as to the actual *form* of the curves.

In the table, $R$ represents the radius of the *rolling* circle, $F$ that of the *fixed* circle.

#### NOMENCLATURE OF TROCHOIDS.

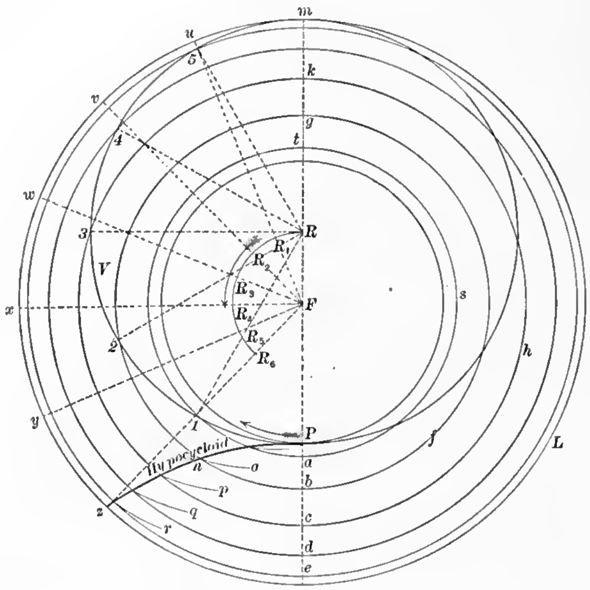
| Position of Tracing or Describing Point. | Circle rolling upon Straight Line. $F=\infty$ | Circle rolling upon circle. External contact. | Circle rolling upon circle. Internal contact. Larger Circle rolling. | Internal contact. Smaller circle rolling. $2R>F$. | Internal contact. Smaller circle rolling. $2R<F$. | Internal contact. Smaller circle rolling. $2R=F$. | Straight Line rolling upon Circle. $R=\infty$ |
|---|---|---|---|---|---|---|---|
| | Ortho-cycloids. | Epitrochoids. | Peritrochoids. | Major Hypotrochoids. | Minor Hypotrochoids. | Medial Hypotrochoids. | Cyclo-orthoids. |
| *On* circumference of rolling circle. | Cycloid. | (a) Epicycloid. | (a) Pericycloid. | (d) Major Hypocycloid. | (d) Minor Hypocycloid. | Straight Hypocycloid. | Involute. |
| *Within* Circumference. | Prolate Trochoid. | (b) Prolate Epitrochoid. | (c) Prolate Peritrochoid. | (e) Major Prolate Hypotrochoid. | (f) Minor prolate Hypotrochoid. | (g) Prolate Elliptical Hypotrochoid. | Prolate Cyclo-orthoid. |
| *Without* Circumference. | Curtate Trochoid. | (c) Curtate Epitrochoid. | (b) Curtate Peritrochoid | (f) Major Curtate Hypotrochoid. | (e) Minor Curtate Hypotrochoid. | (g) Curtate Elliptical Hypotrochoid. | Curtate Cyclo-orthoid. |

177. From the above we see that the prefix *epi* (*over* or *upon*) denotes the curves resulting from external contact; *hypo* (*under*) those of internal contact with smaller circle rolling; while *peri* (*about*) indicates the third possibility as to rolling.

* Re-printed in substance in the Appendix.

178. The construction of these curves is in closest analogy to that of the cycloid. If, for example, we desire a *major hypocycloid* we first draw two circles, $m\,V\,P$, $m\,x\,L$, (Fig. 103), tangent internally, of which the rolling circle has its diameter greater than the radius of the fixed circle. Then, as for the cycloid, if the *tracing-point* is $P$, we divide the semi-circumference $m\,V\,P$ into equal parts, and from the *fixed centre*, $F$, describe circles through the points of division, as those through 1, 2, 3, 4 and 5. These replace the "lines of level" of the cycloid, and may be called *circles of distance*, as they show the varying distances of the point $P$ from $F$, for definite amounts of angular rotation of the former. For if the circle $P\,V\,m$ were simply to rotate about $R$, the point $P$ would reach $m$ during a semi-rotation, and would then be at its maximum distance from $F$. After turning through the equal arcs $P$-1, 1-2, etc., its distances from $F$ would be $F\,a$ and $F\,b$ respectively. If, however, the turning of $P$ about $R$ is due to the rolling of circle $P\,V\,m$ upon the arc $m\,x\,z$, then the *actual position* of $P$, for any amount of turning about $R$, is determined by noting the new position of $R$, due to such rolling, as $R_1$, $R_2$, etc., and from it as a centre cutting the proper circle of distance by an arc of radius $R\,P$.

Fig. 103.




Since the radius of the smaller circle is in this case three-fourths that of the larger, the angle $m\,F\,z$ (135°) at the centre of the latter intercepts an arc, $m\,x\,z$, equal to the 180° arc, $m\,V\,P$, on the smaller circle; for *equal arcs on unequal circles are subtended by angles at the centre which are inversely proportional to the radii.* As a proportion we would have $F\,m : R\,m :: 180° : 135°$. (In an *inverse* proportion between angles and radii, in two circles, the "means" must belong to one circle and the "extremes" to the other).

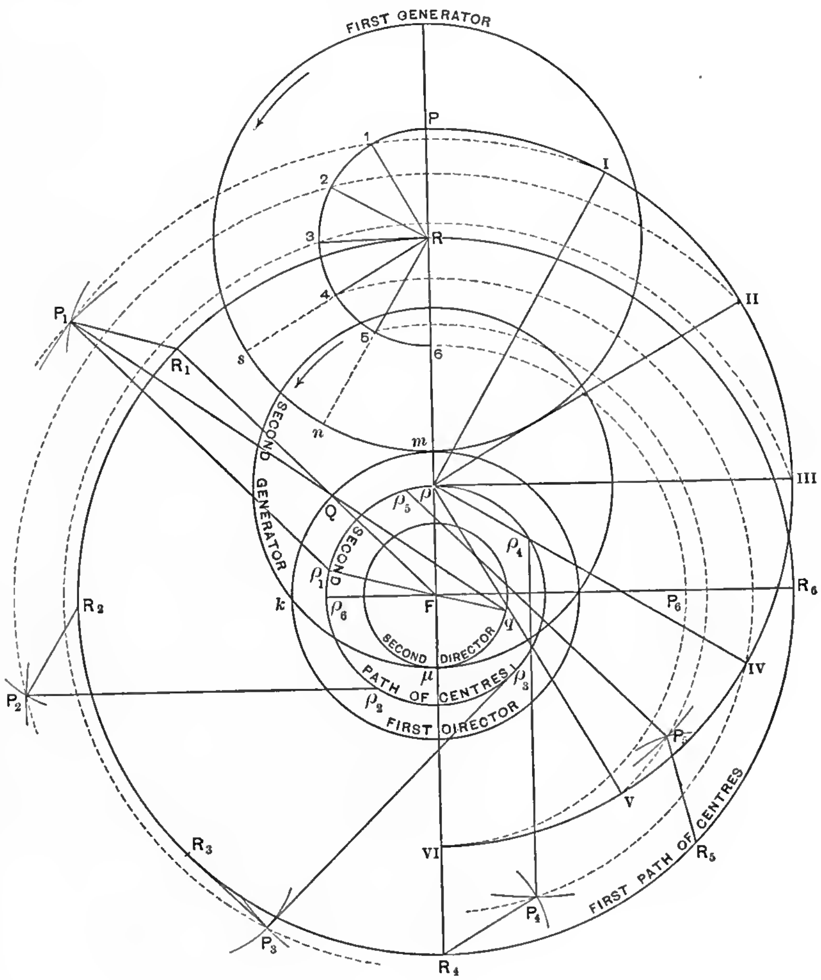
While arc $m\,V\,P$ rolls upon arc $m\,x\,z$ the centre $R$ will evidently move over circular arc $R$---$R_6$. Divide $m\,x\,z$ into as many equal parts as $m\,V\,P$ and draw radii from $F$ to the points of division; these cut the path of centres at the successive positions of $R$. When arc $m\,5$-$4$, for example, has rolled upon its equal $m\,u\,v$ then $R$ will have reached $R_2$; $P$ will have turned about $R$ through angle $P\,R\,2 = m\,R\,4$ and will be at $n$, the intersection of $b\,f\,g$—the circle of distance through 2—by an arc, centre $R_2$, radius $R\,P$. Similarly for other points.

179. *General solution for all trochoidal curves, illustrated by epi- and peri-trochoids.* To trace the path of any point on the circumference of a circle so rolling as to give the *epi-* or *peri-*cycloid requires a construction similar at every step to that of the last article. The same remark applies equally to the path of a point within or beyond the circumference of the rolling circle. This is shown in Fig. 104, before describing which in detail, however, we will summarize the steps for any and all trochoids.

Letting $P$ represent the tracing point, $R$ the *centre* of the *rolling* circle and $F$ that of the *fixed* circle, we draw (1) a circle through $P$, centre $R$; (2) a circle (path of centres) through $R$, centre $F$; (3) ascertain by a proportion (as described in the last article) how many degrees of arc on either circle are equal to the prescribed arc of contact on the other; (4) on the path of centres lay

off—from the initial position of $R$ and in the direction of intended rolling—whatever number of degrees of contact has been assigned or ascertained for the *fixed* circle, and divide this arc by radii from $F$ into *any* number of *equal* parts, to obtain the successive positions of $R$, as $R_1$, $R_2$, etc.; (5) on the circle through $P$ lay off—from the initial position of $P$, and in the direction in which it will move when the assigned rolling occurs—the same number of degrees that have been assigned or calculated as the contact arc of the *rolling* circle, and divide such arc into *the same number of equal parts* that was adopted for the division of the path of centres; (6) through the points of division obtained in the last step draw "circles of distance" with centre $F$, numbering them from $P$; (7) finally, to get the successive positions of $P$, use $RP$ (the "tracing radius") as a constant radius, and cut each circle of distance by an arc from the like-numbered position from $R$, selecting, of course, the right one of the two points in which said curves will always intersect when not tangent.

Fig. 104.




In Fig. 104 the path of the point $P$ is determined (a) as carried by the circle called "first generator," rolling on the exterior of the "first director"; (b) as carried by the "second generator" which rolls on the exterior of the "second director"—which it also encloses. In the first case the resulting curve is a *prolate epitrochoid;* in the second a *curtate peritrochoid;* but such values were taken for the diameters of the circles, that $P$ traced the same curve under either condition of rolling.* These (before reduction with the camera) were 3″ and 2″ for *first generator* and *first director,* respectively.

For the epitrochoid a semi-circle is drawn through $P$ from rolling centre $R;$ similarly with centre $\rho$ for the peritrochoid. Dividing these semi-circles into the same number of equal parts draw next the dotted "circles of distance" through these points, all from centre $F$. The figure illustrates the special case where the two sets of "circles of distance" coincide. The various positions of $P$, as $P_1$, $P_2$, etc., are then located by arcs of radii $RP$ or $\rho P$, struck from the successive positions of $R$ or $\rho$ and intersecting the proper "circle of distance."

* Regarding their double generation refer to the Appendix. In illustrating both methods in one figure it will add greatly to the appearance and also the intelligibility of the drawing if colors are used, red for one construction and blue for the other.

For example, the turning of $P$ through the angle $PR1$ about $R$ would bring $P$ somewhere upon the circle of distance through point 1; but that amount of turning would be due to the rolling of the first generator over the arc $mQ$, which would bring $n$ upon $Q$ and carry $R$ to $R_1$; $P$ would therefore be at $P_1$, at a distance $RP$ from $R_1$, and on the dotted arc through 1. Similarly in relation to $\rho$. When $s$ reached $k$, in the rolling, we would find $P$ at $P_2$.

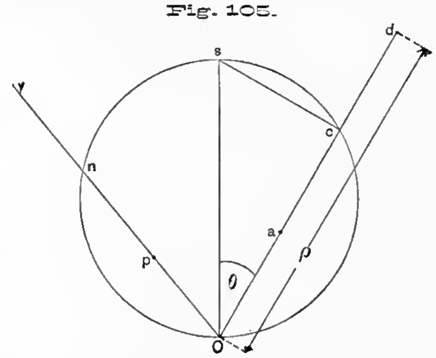
Each position of $P$ is joined with each of the centres from which it could be obtained.

SPECIAL TROCHOIDS.

180. *The Ellipse and Straight Line.* Two circles are called Cardanic* if tangent *internally* and the diameter of one is twice that of the other. If the smaller roll in the larger, all points in the plane of the generator will describe *ellipses* except points *on* the circumference, each of which will move in a straight line—a diameter of the director. Upon this latter property the mechanism known as "White's Parallel Motion" is based, in which a piston-rod is pivoted to a small gear-wheel which rolls on the interior of a toothed annular wheel whose diameter is twice that of the pinion.

181. *The Limaçon and Cardioid.* The Limaçon is a curve whose points may be obtained by drawing random secants through a point on the circumference of a circle, and on each laying off a constant distance, on each side of the second point in which the secant cuts the circle.

Fig. 105.




In Fig. 105 let $Ov$ and $Od$ be random secants of the circle $Ons$; then if $nv$, $np$, $ca$ and $cd$ are each equal to some *constant*, $b$, we shall have $v$, $p$, $a$ and $d$ as four points of a Limaçon. Refer points on the same secant, as $a$ and $d$, to $O$ and the diameter $Os$; we then have $Od = \rho = Oc + cd = 2r\cos\theta + b$, while $Oa = 2r\cos\theta - b$; hence the polar equation is $\rho = 2r\cos\theta \pm b$.

When $b = 2r$ the Limaçon becomes a *Cardioid*.† (See Fig. 106).

182. All Limaçons, general and special, may be generated either as epi- or peri-trochoidal curves: as *epi*-trochoids the generator and director must have *equal* diameters, any point *on* the circumference of the generator then tracing a Cardioid, while any point on the radius (or radius produced) describes a Limaçon; as *peri*-trochoids the larger of a pair of Cardanic circles must roll on the smaller, the Cardioid and Limaçon then resulting, as before, from the motion of points respectively *on* the circumference of the generator, or *within* or *without* it.

183. In Fig. 106 the Cardioid is obtained as an epicycloid, being traced by point $P$ during one revolution of the generator $PHm$ about an equal directing circle $msO$.

As a Limaçon we may get points of the Cardioid, as $y$ and $z$, by drawing a secant through $O$ and laying off $sy$ and $sz$ each equal to $2r$.

184. *The Limaçon as a Trisectrix.* Three famous problems of the ancients were the squaring of the circle, the duplication of the cube and the trisection of an angle. Among the interesting curves invented by early mathematicians for the purpose of solving one or the other of these problems were the Quadratrix and Conchoid, whose construction is given later in this chapter; but it has been found that certain trochoids may as readily be employed for trisection, among them the Limaçon of Fig. 106, frequently called the *Epitrochoidal Trisectrix.*

When constructed as a Limaçon we find points as $G$ and $X$, on any secant $RX$ of the circle called "path of centres," by making $SX$ and $SG$ each equal to the radius of that circle.

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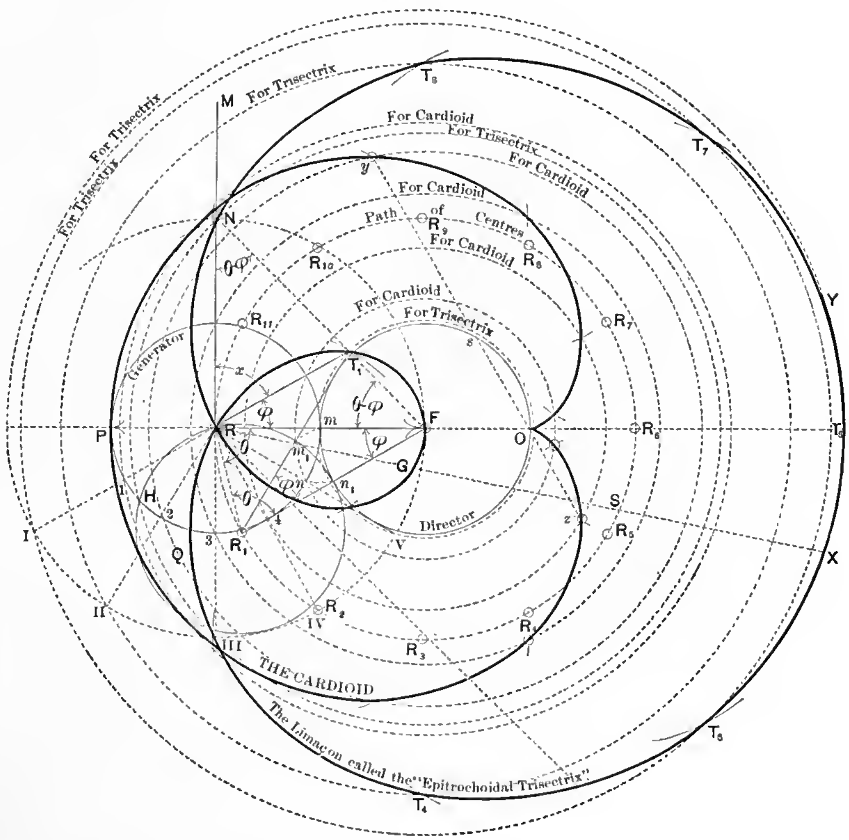
*Term due to Reuleaux, and based upon the fact that Cardano (16th century) was probably the first to investigate the paths described by points during their rolling.

†From *Cardis*, the Latin for *heart.*

185. *To trisect an angle*, as $MRF$, by means of this epitrochoid, bisect one side of the angle, as $FR$, at $m$; use $mR$ and $mF$ as radii for generator and director respectively of an epitrochoid having a tracing radius, $RF$, equal to *twice* that of the generator. Make $RN = RF$ and draw $NF$; this will cut the Limaçon $FT_1RQ$ (traced by point $F$ as carried by the given generator) in a point $T_1$. The angle $T_1RF$ will then be one-third of $NRF$, which may be proved as follows: $F$ reaches $T_1$ by the rolling of arc $mn$ on arc $mn_1$. These arcs are subtended by equal angles, $\phi$, the circles being equal. During this rolling $R$ reaches $R_1$, bringing $RF$ to $R_1T_1$. In the triangles $T_1R_1F$ and $RFR_1$ the side $FR_1$ is common, angles $\phi$ equal, and side $R_1T_1$ equal to side $RF$; the line $RT_1$ is therefore parallel to $R_1F$, whence angle $T_1RF$ must also equal $\phi$. In the triangle $RFR_1$ we denote by $\theta$ the angles opposite the equal sides $RF$ and $R_1F$; then $2\theta + \phi = 180°$, or $\theta = \frac{180° - \phi}{2}$. In triangle

Fig. 106.




$NRF$ we have the angle at $F$ equal to $\theta - \phi$, and $2(\theta - \phi) + x + \phi = 180°$, which gives $x = 2\phi$ by substituting the value of $\theta$ from the previous equation.

186. *The Involute.* As the opposite extreme of a circle rolling on a straight line we may have the latter rolling on a circle. In this case the *rolling* circle has an *infinite radius*. A point on the straight line describes a curve called the *involute*. This would be the path of the end of a thread if the latter were in tension while being unwound from a spool.

In Fig. 107 a rule is shown, tangent at $u$ to a circle on which it is supposed to roll. Were a pencil-point inserted in the centre of the circle at $j$ (which is on the line $ux$ produced) it would trace the involute. When $j$ reaches $a$ the rule will have had rolling contact with the base circle over an arc $uts\text{---}a$ whose length equals line $uxj$. Were $a$ the initial point we would obtain, $b$, $c$,

etc., by making *tangent* $mb = arc\ ma$; *tangent* $nc = arc\ na$. Each tangent thus equals the arc from the initial point to the point of tangency.

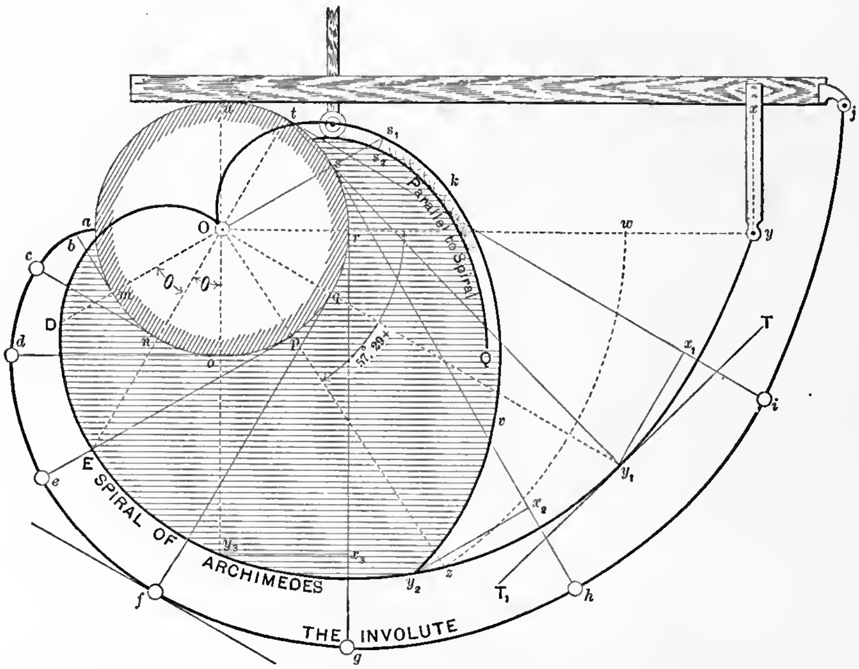
187. The circle from which the involute is derived or *evolved* is called the *evolute.* Were a hexagon or other figure to be taken as an evolute a corresponding involute could be derived; but the name "involute," unqualified, is understood to be that obtained from a circle.

From the law of formation of the involute the rolling line is in all its positions a normal to the curve; the point of tangency on the evolute is an instantaneous centre, and a tangent at any point, as $f$, is a perpendicular to the tangent, $fq$, from $f$ to the base circle.

Like the cycloid, the involute is a correct working outline for the teeth of gear-wheels; and gears manufactured on the involute system are to a considerable degree supplanting other forms.

A surface known as the *developable helicoid* (see Figs. 209 and 270) is formed by moving a line

Fig. 107.




so as to be always *tangent* to a given helix. It is interesting in this connection to notice that any plane perpendicular to the axis of the helix would cut such a surface in a pair of involutes.*

188. *The Spiral of Archimedes.* This curve is generated by a point having a uniform motion around a fixed point—the *pole*—combined with uniform motion toward or from it.

In Fig. 107, with $O$ as the pole, if the angles $\theta$ are equal, and $OD$, $OE$ and $Oy_3$ are in arithmetical progression, then the points $D$, $E$ and $y_3$ are points of an Archimedean Spiral.

This spiral can be trochoidally generated, simultaneously with the involute, by inserting a pencil point at $y$ in a piece carried by—and at right angles with—the rule, the point $y$ being at a distance,

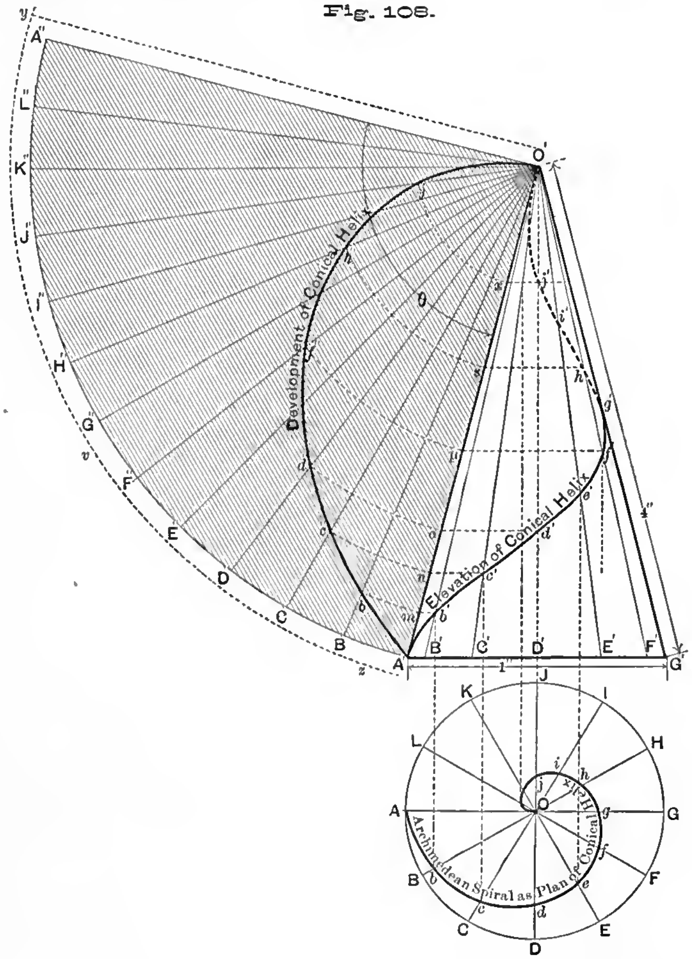
*The day of writing the above article the following item appeared in the New York *Evening Post:* "Visitors to the Royal Observatory, Greenwich, will hereafter miss the great cylindrical structure which has for a quarter century and more covered the largest telescope possessed by the Observatory. Notwithstanding its size the Astronomer Royal has now procured through the Lords Commissioners a telescope more than twice as large as the old one.... The optical peculiarities embodied in the new instrument will render it one of the three most powerful telescopes at present in existence.... The peculiar architectural feature of the building which is to shelter the new telescope is that its dome, of thirty-six feet diameter, will surmount a tower having a diameter of only thirty-one feet. Technically, the form adopted is the surface generated by the revolution of an *involute of a circle.*"

$xy$, from the contact-edge of the rule, equal to the radius $Os$ of the base circle of the involute; for after the rolling of $ux$ over an arc $ut$ we shall have $tx_1$ as the portion of the rolling line between $x$ and the point of tangency, and $xy$ will have reached $x_1y_1$. If the rolling be continued $y$ will evidently reach $O$. We see that $Oy=ux$, and $Oy_1=tx_1$; but the lengths $ux$ and $tx_1$ are proportional to the angular movement of the rolling line about $O$, and as the spiral may be defined as that curve in which the length of a radius vector is directly proportional to the angle through which it has turned about the pole, the various positions of $y$ are evidently points of such a curve.

189. *A Tangent to the Spiral of Archimedes.* Were the pole, $O$, given, and a portion only of the spiral, we could draw a tangent at any point, $y_1$, by determining the circle on which the spiral could be trochoidally generated, then the instantaneous centre for the given position of the tracing-point, whence the normal and tangent would be derived in the usual way. The radius $Ot$ of the base circle would equal $wy$—the difference between two radii vectores $Oy$ and $Oz$ which include an angle of 57° 29+, (the angle which at the centre of a circle subtends an arc equal to the radius). The instantaneous centre, $t$, would be the extremity of that radius which was perpendicular to $Oy_1$. The *normal* would be $ty_1$, and the *tangent* $TT_1$ perpendicular to it.

190. The spiral of Archimedes is the right section of an oblique helicoid. (Art. 357). It is also the proper outline for a cam to convert uniform rotary into uniform rectilinear motion, and when combined with an equal and opposite spiral gives the well-known form called the *heart-cam.* As usually constructed the acting curve is not the true spiral, but a curve whose points are at a constant distance from the theoretical outline equal to the radius of the friction-roller which is on the end of the piece to be raised. $Qs_2$ (Fig. 107) is a small portion of such a "parallel curve."

Fig. 108.




191. If a point travel on the surface of a cone so as to combine a uniform motion around the axis with a uniform motion toward the vertex it will trace a *conical helix,* whose orthographic projection on the plane of the base will be a spiral of Archimedes.

In Fig. 108 a top and front view are given of a cone and helix. The shaded portion is the *development* of the cone, that is, the area equal to the convex surface, and which—if rolled up—would form the cone. To obtain the development draw an arc $A'G''A''$ of radius equal to an element. The convex surface of the cone will then be represented by the sector $A'O'A''$, whose angle $\theta$ may be found by the proportion $OA:O'A'::\theta:360°$, since the arc $A'G''A''$ must equal the entire circumference of the cone's base.

The student can make a paper model of the cone and helix by cutting out a sector of a circle,

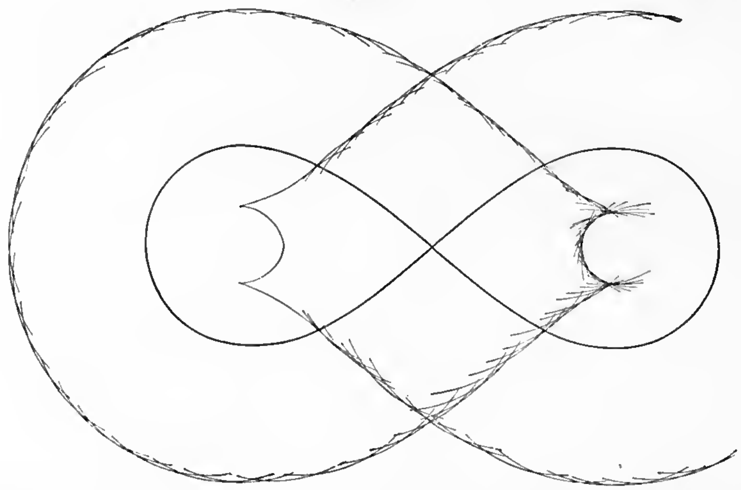
making allowance for an overlap on which to put the mucilage, as shown by the dotted lines $O'y$ and $y\,v\,z$ in the figure.

The *development* of a conical helix is the same kind of spiral as its orthographic projection.

PARALLEL CURVES.

192. A *parallel curve* is one whose points are at a constant normal distance from some other curve. Parallel curves have not the same mathematical properties as those from which they are derived, except in the case of a circle; this can readily be seen from the cam figure under the last heading, in which a point, as $S_1$, of the true spiral, is located on a line from $O$ which is by no means in the direction of the *normal* to the curve at $S_1$, upon which lies the point $S_2$ of the parallel curve.

Fig. 109.



Instead of actually determining the normals to a curve and on each laying off a constant distance, we may draw many arcs of constant radius, having their centres on the original curve; the desired parallel will be tangent to all these arcs.

In strictly mathematical language a *parallel curve* is the *envelope* of a circle of constant radius whose centre is on the original curve. We may also define it as the locus of consecutive intersections of a system of equal circles having their centres on the original curve.

If on the *convex* side of the original the parallel will resemble it in form, but if *within*, the two may be totally dissimilar. This is well illustrated in Fig. 109, in which the parallel to a Lemniscate is shown.

The student will obtain some interesting results by constructing the parallels to ellipses, parabolas and other plane curves.

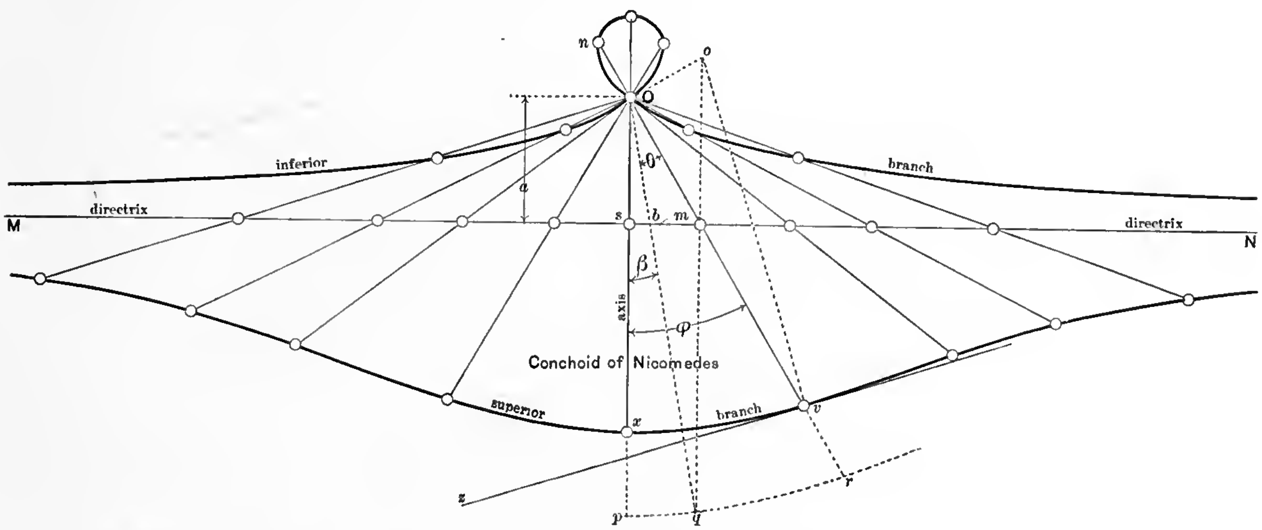
THE CONCHOID OF NICOMEDES.

193. The *Conchoid*, named after the Greek word for *shell*,* may be obtained by laying off a constant length on each side of a given line $MN$ (the *directrix*) upon radials through a fixed point or *pole*, $O$ (Fig. 110). If $mv = mn = sx$ then $v$, $n$ and $x$ are points of the curve. Denote by $a$ the distance of $O$ from $MN$, and use $e$ for the constant length to be laid off; then if $a < e$ there will be a loop in that branch of the curve which is nearest the pole,—the *inferior* branch. If $a = e$ the curve has a point or *cusp* at the pole. When $a > e$ the curve has an undulation or wave-form towards the pole.

* A series of curves much more closely resembling those of a shell can be obtained by tracing the paths of points on the piston-rod of an oscillating cylinder. See Arts. 157 and 158 for the principles of their construction.

$Ov = c + Om$; $On = c - Om$; we may therefore express the relation to $O$ of points on the curve by the equation $\rho = c \pm O\,m = c \pm a \sec \phi$.

Fig. 110.




194. Mention has already been made (Art. 184) of the fact that this was one of the curves invented in part for the purpose of solving the problem of the *trisection of an angle*. Were $m\,O\,x$ (or $\phi$) the angle to be trisected we would first draw $p\,q\,r$, the superior branch of a conchoid having the constant, $c$, equal to twice $O\,m$. A parallel from $m$ to the axis will intersect the curve at $q$; the angle $p\,O\,q$ will then be one-third of $\phi$: for since $b\,q = 2\,O\,m$ we have $m\,q = 2\,O\,m \cos\beta$; also $m\,q : O\,m :: \sin\theta : \sin\beta$; hence $2\,O\,m \cos\beta : O\,m :: \sin\theta : \sin\beta$, whence $\sin\theta = 2 \sin\beta \cos\beta = \sin 2\beta$ (from known trigonometric relations). The angle $\phi$ is therefore equal to twice $\beta$, which makes the latter one-third of angle $\phi$.

195. *To draw a tangent and normal* at any point $r$ we find the instantaneous centre $o$ on the principle that it is at the intersection of normals to the paths of two moving points of a line, the distance between said points remaining constant. In tracing the curve the motion of $O$ (on $O\,v$) is —at the instant considered—in the direction $O\,v$; $O\,o$ is therefore the normal. The point $m$ of $O\,v$ is at the same moment moving along $M\,N$, for which $m\,o$ is the normal. Their intersection $o$ is then the instantaneous centre, and $o\,v$ the normal to the conchoid, with $v\,z$ perpendicular to $o\,v$ for the desired tangent.

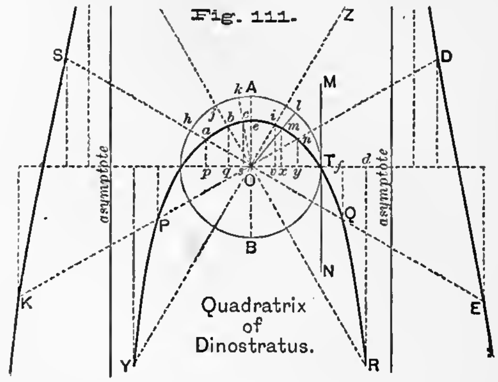
196. This interesting curve may be obtained as a plane section of one of the higher mathematical surfaces. If two non-intersecting lines—one vertical, the other horizontal—be taken as guiding lines or *directrices* of the motion of a third straight line whose inclination to a horizontal plane is to be constant, then horizontal planes will cut *conchoids* from the surface thus generated, while every plane parallel to the directrices will cut *hyperbolas*. From the nature of its plane sections this surface is called the *Conchoidal Hyperboloid*. (See Fig. 219).

### THE QUADRATRIX OF DINOSTRATUS.

197. In Fig. 111 let the radius $O\,T$ rotate uniformly about the centre; simultaneously with its movement let $M\,N$ have a uniform motion parallel to itself, reaching $A\,B$ at the same time with radius $O\,T$; the locus of the intersection of $M\,N$ with the radius will be the *Quadratrix*. Points

exterior to the circle may be found by prolonging the radii while moving $MN$ away from $AB$. As the intersection of $MN$ with $OB$ is at infinity, the former becomes an asymptote to the curve as often as it moves from the centre an additional amount equal to the diameter of the circle; the number of branches of the Quadratrix may therefore be infinite. It may be proved analytically that the curve crosses $OA$ at a distance from $O$ equal to $2r \div \pi$.

Fig. 111.




198. *To trisect an angle*, as $TOa$, by means of the Quadratrix, draw the ordinate $ap$, trisect $pT$ by $s$ and $x$ and draw $sc$ and $xm$; radii $Oc$ and $Om$ will then divide the angle as desired: for by the conditions of generation of the curve the line $MN$ takes three equidistant parallel positions while the radius describes three equal angles.

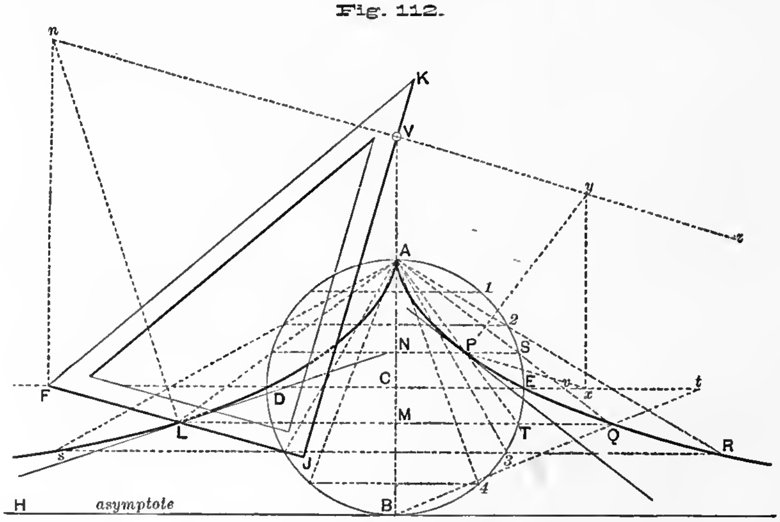
THE CISSOID OF DIOCLES.

199. This curve was devised for the purpose of obtaining two mean proportionals between two given quantities, by means of which the duplication of the cube might be effected.

The name was suggested by the Greek word for *ivy*, since "the curve appears to mount along its asymptote in the same manner as that parasite plant climbs on the tall trunk of the pine."*

This was one of the first curves invented after the discovery of the conic sections. Let $C$ (Fig. 112) be the centre of a circle, $ACE$ a right angle, $NS$ and $MT$ *any* pair of ordinates *parallel to*

Fig. 112.




*and equidistant from CE;* then a secant from $A$ through the extremity of either ordinate will meet the other ordinate in a point of the cissoid. $AT$ and $NS$ give $P$; $AS$ and $MT$ give $Q$.

The tangent to the circle at $B$ will be an asymptote to the curve.

It is a somewhat interesting coincidence that the area between the cissoid and its asymptote is the same as that between a cycloid and its base, viz., three times that of the circle from which it is derived.

200. Sir Isaac Newton devised the following method of obtaining a cissoid by continuous motion: Make $AV = AC$; then move a right-angled triangle, of base $= VC$, so that the vertex $F$ travels along

* Leslie. *Geometrical Analysis.* 1821.

the line $DE$ while the edge $JK$ always passes through $V$; then the middle point, $L$, of the base $FJ$, will trace a cissoid. This construction enables us readily to get the instantaneous centre and a tangent and normal; for $Fn$ is normal to $FC$—the path of $F$, while $nV$ is normal to the motion of $J$ toward $JV$; the instantaneous centre $n$ is therefore at the intersection of these normals. For any other point as $P$ we apply the same principle thus: With radius $AC$ and centre $P$ obtain $x$; draw $Px$, then $Vz$ parallel to it; a vertical from $x$ will meet $Vz$ at the instantaneous centre $y$, whence the normal and tangent result in the usual way. The point $y$ does not necessarily fall on $nV$.

Since $nV$ and $FJ$ are perpendicular to $JV$ they are parallel. So also must $Vz$ be parallel to $Px$, regardless of where $P$ is taken.

201. Two quantities $m$ and $n$ will be mean proportionals between two other quantities $a$ and $b$ if $m^2 = na$ and $n^2 = mb$; that is, if $m^3 = a^2 b$ and if $n^3 = ab^2$.

If $b = 2a$ we will find, from the relation $m^3 = a^2 b$, that $m$ will be the edge of a cube whose volume equals $2a^3$.

To get two mean proportionals between quantities, $r$ and $b$, make the smaller, $r$, the radius of a circle from which derive a cissoid. Were $APR$ the derived curve we would then make $Ct$ equal to the second quantity, $b$, and draw $Bt$, cutting the cissoid at $Q$. A line $AQ$ would cut off on $Ct$ a distance $Cv$ equal to $m$, one of the desired proportionals; for $m^3$ will then equal $r^2 b$, as may be thus shown by means of similar triangles:

$$Cv : MQ :: CA : MA \quad \text{whence} \quad Cv^3 = \frac{r^3 . MQ^3}{MA^3} \qquad (1)$$

$$Ct : MQ :: CB : BM \quad \text{"} \quad Ct = \frac{r . MQ}{BM} \qquad (2)$$

$$MQ : MA :: SN : AN :: \sqrt{AN . BN} : AN, \text{ whence } MQ = \frac{MA\sqrt{AN . BN}}{AN} \qquad (3)$$

$$\text{From (2) we have } MQ = \frac{BM(Ct = b)}{r} \qquad (4)$$

$$\text{" (3) " " } MQ^2 = \frac{MA^2(AN . BN)}{AN^2} \qquad (5)$$

Replacing $MQ^3$ in equation (1) by the product of the second members of equations (4) and (5) gives $Cv^3$ (i.e., $m^3$) $= r^2 b$.

By interchanging $r$ and $b$ we obtain $n$, the other mean proportional; or it might be obtained by constructing similar triangles having $r$, $b$ and $m$ for sides.

## THE TRACTRIX.

202. The *Tractrix* is the involute of the curve called the *Catenary* (Art. 214) yet its usual construction is based on the fact that if a series of tangents be drawn to the curve, the portions of such tangents between the points of tangency and a given line will be of the same length; or, in other words, the intercept on the tangent, between the directrix and the curve, will be constant. A practical and very close approximation to the theoretical curve is obtained by taking a radius $QR$ (Fig. 113) and with a centre $a$, a short distance from $R$ on $QR$, obtaining $b$, which is then joined with $a$. On $ab$ a centre $c$ is similarly taken for another arc of the same radius, whence $cd$ is obtained. A sufficient repetition of this process will indicate the curve by its enveloping tangents, or a curve may actually be drawn tangent to all these lines. Could we take $a$, $b$, $c$, etc., as mathematically *consecutive points* the curve would be theoretically exact. The line $QS$ is an asymptote to the curve.

The area between the completed branch $RPS$ and the lines $QR$ and $QS$ would be equal to a quadrant of the circle on radius $QR$.

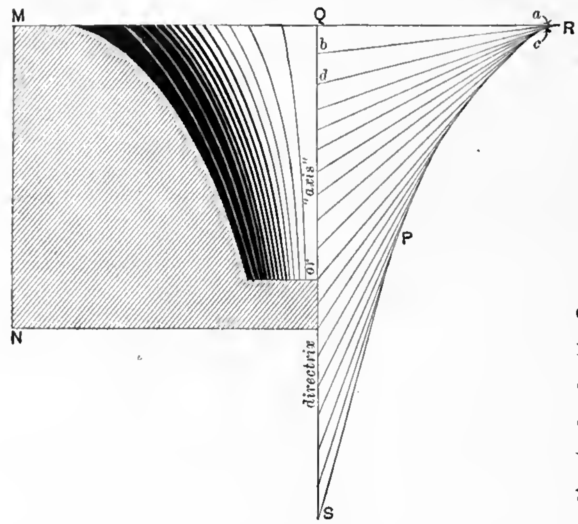



Fig. 113.

203. The surface generated by revolving the tractrix about its asymptote has been employed for the foot of a vertical spindle or shaft, and is known as Schiele's Anti-Friction Pivot. The step for such a pivot is shown in sectional view in the left half of the figure. Theoretically, the amount of work done in overcoming friction is the same on all equal areas of this surface.

In the case of a bearing of the usual kind, for a cylindrical spindle, although the *pressure* on each square inch of surface would be constant, yet, as unit areas at different distances from the centre would pass over very different amounts of space in one revolution, the wear upon them would be necessarily unequal. The *rationale* of the tractrix form will become evident from the following consideration: If about to split a log, and having a choice of wedges, any boy would choose a thin one rather than one with a large angle, although he might not be able to prove by graphical statics the exact amount of advantage the one would have over the other. The theory is very simple, however, and the student may profitably be introduced to it. Suppose a ball, $c$, (Fig. 114) struck at the same instant by two others, $a$ and $b$, moving at rates of six and eight feet a second respectively. On $ac$ and $bc$ prolonged take $ce$ and $ch$ equal, respectively, to *six* and *eight* units of some scale; complete the parallelogram having these lines as sides; then it is a well-known principle in mechanics* that $cd$ — the diagonal of this parallelogram — will not only represent the *direction* in which the ball $c$ will move, but also the *distance* — in feet, to the scale chosen — it will travel in one second. Evidently, then, to *balance* the effect of balls $a$ and $b$ upon $c$, a fourth would be necessary, moving from $d$ toward $c$ and traversing $dc$ in the same second that $a$ and $b$ travel, so that impact of all would occur simultaneously. These forces would be represented in direction and magnitude (to some scale) by the shaded triangle $c'd'e'$, which illustrates the very important theorem that if the three sides of a triangle — taken like $c'e'$, $e'd'$, $d'c'$, in such order as to bring one back to the initial vertex mentioned — represent in *magnitude* and *direction* three forces acting on one point, then these forces are balanced.

Fig. 114.

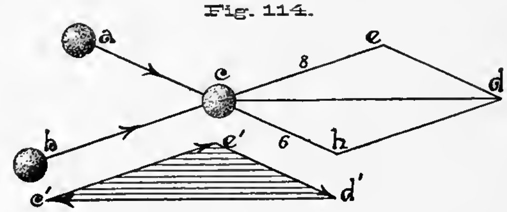
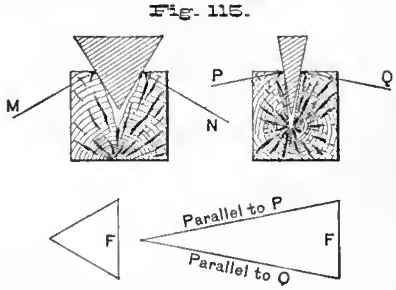



Fig. 115.




Constructing now a triangle of forces for a broad and thin wedge, (Fig. 115) and denoting the force of the supposed *equal blows* by $F$ in each triangle, we see that the pressures are greater for the thin wedge than for the other; that is, the less the inclination to the vertical the greater the pressure. A pivot so shaped that as the *pressure* between it and its step *increased* the *area* to be traversed *diminished* would therefore, theoretically, be the ideal; and the rate of change of curvature of the tractrix, as its generating point approaches the axis, makes it, obviously, the correct form.

*For a demonstration the student may refer to Rankine's *Applied Mechanics*, Art. 51.

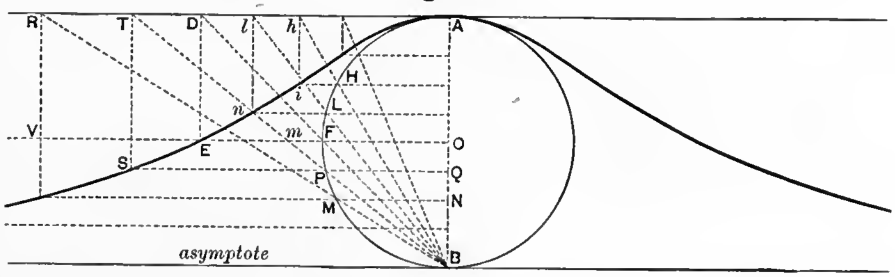
204. Navigator's charts are usually made by *Mercator's projection* (so-called, not being a *projection* in the ordinary sense, but with the extended signification alluded to in the remark in Art. 2). Maps thus constructed have this advantageous feature, that *rhumb lines* or *Loxodromics*—the curves on a sphere that cut all meridians at the same angle—are represented as *straight lines*, which can only be the case if the meridians are indicated by *parallel* lines. The law of convergence of meridians on a sphere is, that the length of a degree of longitude at any latitude equals that of a degree on the equator multiplied by the cosine (see foot-note, p. 31) of the latitude; when the meridians are made *non-convergent* it is, therefore, manifestly necessary that the distance apart of originally equidistant parallels of latitude must increase at the same rate; or, otherwise stated, as on Mercator's chart degrees of longitude are all made equal, regardless of the latitude, the constant length representative of such degree bears a varying ratio to the actual arc on the sphere, being greater with the increase in latitude; but the greater the latitude the less its cosine or the greater its secant; hence lengths representative of degrees of latitude will increase with the secant of the latitude. Tables have been constructed giving the increments of the secant for each minute of latitude; but it is an interesting fact that they may be derived from the Tractrix thus: Draw a circle with radius $QR$, centre $Q$ (Fig. 113); estimate latitude on such circle from $R$ upward; the intercept on $QS$ between consecutive tangents to the Tractrix will be the increment for the arc of latitude included between parallels to $QS$, drawn through the points of contact of said pair of tangents.*

On map construction the student is referred to Chapter XII, or to Craig's *Treatise on Projections.*

THE WITCH OF AGNESI.

205. If on any line $SQ$, perpendicular to the diameter of a circle, a point $S$ be so located that $SQ:AB::PQ:QB$ then $S$ will be a point of the curve called the *Witch of Agnesi.* Such point is evidently on the ordinate $PQ$ prolonged, and vertically below the intersection $T$ of the tangent at $A$ by the secant through $P$.

Fig. 116.




The point $E$, at the same level as the centre $O$, is a diameter's distance from the latter.

The tangent at $B$ is an asymptote to the curve.

The area between the curve and its asymptote is four times that of the circle involved in its construction.

The *Witch*, also called the *Versiera*, was devised by Donna Maria Gaetana Agnesi, a brilliant Italian lady who was appointed in 1750, by Pope Benedict XIV, to the professorship of mathematics and philosophy in the University of Bologna.

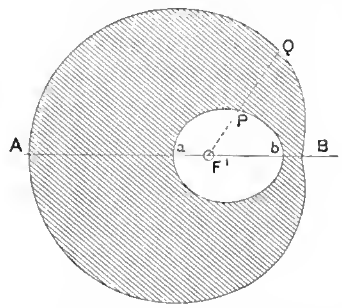
THE CARTESIAN OVAL.

206. This curve, also called simply a *Cartesian*, after its investigator, Descartes, has its points connected with two foci, $F'$ and $F''$, by the relation $m\rho' \pm n\rho'' = kc$, in which $c$ is the distance between the foci, while $m$, $n$ and $k$ are constant factors.

*Leslie. *Geometrical Analysis.* Edinburgh, 1821.

Salmon states that we owe to Chasles the proof that a third focus may be found, sustaining the same relation, and expressed by an equation of similar form. (See Art. 209).

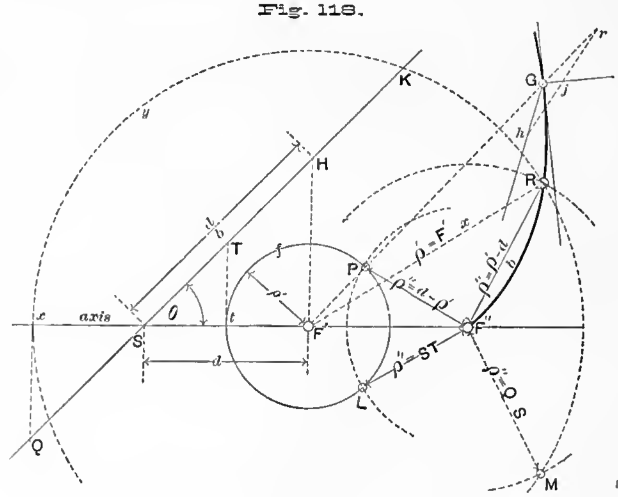
Fig. 117.




The Cartesian is symmetrical with respect to the *axis*—the line joining the foci.

207. To construct the curve from the first equation we may for convenience write $m\rho' \pm n\rho'' = kc$ in the form $\rho' \pm \frac{n}{m}\rho'' = \frac{kc}{m}$; or by denoting $\frac{n}{m}$ by $b$ and $\frac{kc}{m}$ by $d$ it takes the yet more simple form $\rho' \pm b\rho'' = d$. Then $\rho''$ will have two values according as the positive or negative sign is taken, being respectively $\frac{d-\rho'}{b}$ and $\frac{\rho'-d}{b}$; the former is for points on the *inner* of the *two* ovals that constitute a complete Cartesian, while the latter gives points on the *outer* curve.

Fig. 118.




To obtain $\rho'' = \frac{d-\rho'}{b}$ take $F'$ and $F''$ (Fig. 118) as foci; $F'S = d$; $SK$ at some random acute angle $\theta$ with the axis, and make $SH = \frac{d}{b}$; that is, make $F'S : SH :: b : 1$. Then from $F'$ draw an arc $tfP$, of radius less than $d$, and cut it at $P$ by an arc from centre $F''$, radius $ST$, $Tt$ being a parallel to $F'H$; then $P$ is a point of the inner oval; for $St = d - \rho'$, and $ST = \rho''$; therefore $\rho'' : d - \rho' :: \frac{d}{b} : d$, whence $\rho'' = \frac{d-\rho'}{b}$.

208. If an arc $xyK$ be drawn from $F'$, with radius, $F'x$, *greater than* $d$, we may find the second value of $\rho''$, viz., $\frac{\rho'-d}{b}$, by drawing $xQ$ parallel to $F'H$ to meet $HS$ prolonged; for $QS$ will equal $\frac{\rho'-d}{b}$, in which $\rho' = F'x$. Again using $F''$ as a centre, and a radius $QS = \rho''$, gives points $R$ and $M$ of the larger oval.

The following are the values for the focal radii to the four points where the ovals cut the axes. (See Fig. 117).

$$\text{For } A,\quad \rho'' = \frac{\rho'-d}{b} = c + \rho' \quad \text{whence} \quad \rho' = F'A = \frac{d+bc}{1-b}$$

$$\text{“}\quad a,\quad \rho'' = \frac{d-\rho'}{b} = c + \rho' \quad \text{“} \quad \rho' = F'a = \frac{d-bc}{1+b}$$

$$\text{“}\quad B,\quad \rho'' = \frac{\rho'-d}{b} = c - \rho' \quad \text{“} \quad \rho' = F'B = \frac{d+bc}{1+b}$$

$$\text{“}\quad b,\quad \rho'' = \frac{d-\rho'}{b} = c - \rho' \quad \text{“} \quad \rho' = F'b = \frac{d-bc}{1-b}$$

The construction-arcs for the outer oval must evidently have radii *between* the values of $\rho'$ for $A$ and $B$ above; and for the inner oval *between* those of $a$ and $b$.

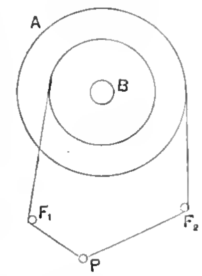
The numerical values from which Fig. 118 was constructed were $m = 3$; $n = 2$; $c = 1$; $k = 3$.

209. *The Third Focus.* Fig. 118 illustrates a special case, but, in general, the method of finding a third focus $F'''$ (not shown) would be to draw a random secant $F'r$ through $F'$, and note the points $P$ and $G$ in which it cuts the ovals—these to be taken on the same side of $F'$, as two other points of intersection are possible; a circle through $P$, $G$ and $F''$ would cut the axis in the new focus sought. Then denoting by $C$ the distance $F' F'''$, we would find the factors of the original equation appearing in a new order; thus, $k\rho' \pm n\rho''' = mC$, which—for purposes of construction—may be written $\rho' \pm b'\rho''' = d'$.

If obtained from the foci $F''$ and $F'''$ the relation would be $m\rho''' - k\rho'' = \pm nC'$, in which $C'$ equals $F'' F'''$. Writing this in the form $\rho''' - B\rho'' = \pm D$ we have the following interesting cases: (a) an *ellipse* for $D$ positive and $B = -1$; (b) an *hyperbola* for $D$ positive and $B = +1$; (c) a *limaçon* for $D = C' B$; (d) a *cardioid* for $B = +1$ and $D = C'$.

210. The following method of drawing a Cartesian by continuous motion was devised by Prof. Hammond: A string is wound, as shown, around two pulleys turning on a common axis; a pencil at $P$ holds the string taut around smooth pegs placed at random at $F_1$ and $F_2$; if the wheels be turned with the same angular velocity, and the pencil does not slip on the string, it will trace a Cartesian having $F_1$ and $F_2$ as foci.*

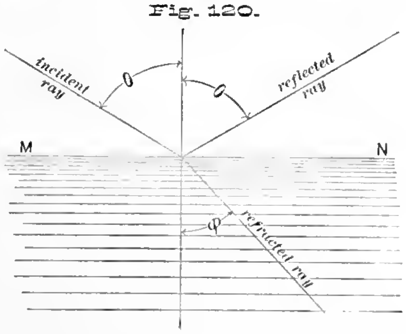
Fig. 119.




If the pulleys are *equal* the Cartesian will become an *ellipse;* if both threads are wound *the same way* around *either one* of the wheels the resulting curve will be an *hyperbola.*

211. It is a well-known fact in Optics that the incident and reflected ray make equal angles with the normal to a reflecting surface. If the latter is *curved* then each reflected ray cuts the one next to it, their consecutive intersections giving a curve called a *caustic by reflection.* Probably all have occasionally noticed such a curve on the surface of the milk in a glass, when the light was properly placed. If the reflecting curve is a *circle* the caustic is the *evolute of a limaçon.*

Fig. 120.




In passing from one medium *into* another, as from air into water, the deflection which a ray of light undergoes is called *refraction,* and for the same media the ratio of the *sines* of the angles of incidence and refraction ($\theta$ and $\phi$, Fig. 120) is constant. The consecutive intersections of *refracted* rays give also a *caustic,* which, for a *circle,* is the *evolute* of a *Cartesian Oval.* The proof of this statement† involves the property upon which is based the *most convenient method of drawing a tangent to the Cartesian,* viz., that the normal at any point divides the angle between the focal radii into parts whose sines are proportional to the factors of those radii in the equation. If, then, we have obtained a point $G$ on the outer oval from the relation $m\rho' \pm n\rho'' = kc$, we may obtain the tangent at $G$ by laying off on $\rho'$ and $\rho''$ distances proportional to $m$ and $n$, as $Gr$ and $Gh$, Fig. 118, then bisecting $rh$ at $j$ and drawing the normal $Gj$, to which the desired tangent is a perpendicular.

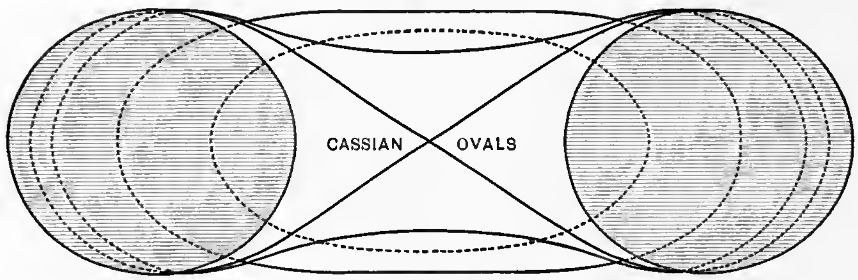
At a point on the inner oval the distance would not be laid off on a focal radius *produced*, as in the case illustrated.

* *American Journal of Mathematics*, 1878. † Salmon. *Higher Plane Curves.* Art. 117.

## CASSIAN OVALS.

212. In the *Cassian Ovals* or *Ovals of Cassini* the points are connected with two foci by the relation $\rho'\rho''=k^2$, i.e., the *product* of the focal radii is equal to some perfect square. These curves have already been alluded to in Art. 114 as plane sections of the annular torus, taken parallel to its axis.

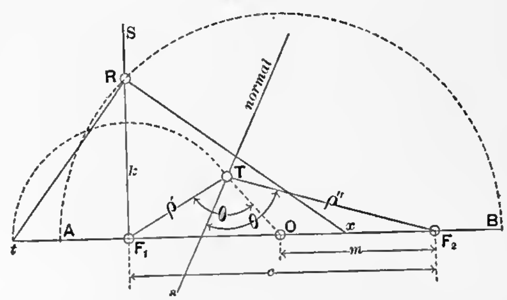
Fig. 121.




In Art. 158 one form—the Lemniscate—receives special treatment. For it the constant $k^2$ must equal $m^2$, the square of half the distance between the foci. When $k$ is less than $m$ the curve becomes two separate ovals.

213. The general construction depends on the fact that in any semicircle the square of an ordinate equals the product of the segments into which it divides the diameter. In Fig. 122 take $F_1$ and $F_2$ as the foci, erect a perpendicular $F_1S$ to the axis $F_1F_2$, and on it lay off $F_1R$ equal to the constant, $k$. Bisect $F_1F_2$ at $O$ and draw a semicircle of radius $OR$. This cuts the axis at $A$ and $B$, the extreme points of the curve; for $k_2=F_1A\times F_1B$. Any other point $T$ may be obtained by drawing from $F_1$ a circular arc of radius $F_1t$ greater than $F_1A$; draw $tR$, then $Rx$ perpendicular to it; $xF_1$ will then be the $\rho''$, and $F_1t$ the $\rho'$, for four points of the curve, which will be at the intersection of arcs struck from $F_1$ and $F_2$ as centres and with those radii.

Fig. 122.




To get a normal at any point $T$ draw $OT$, then make angle $F_2Ts=\theta=F_1TO$; $Ts$ will be the desired line.

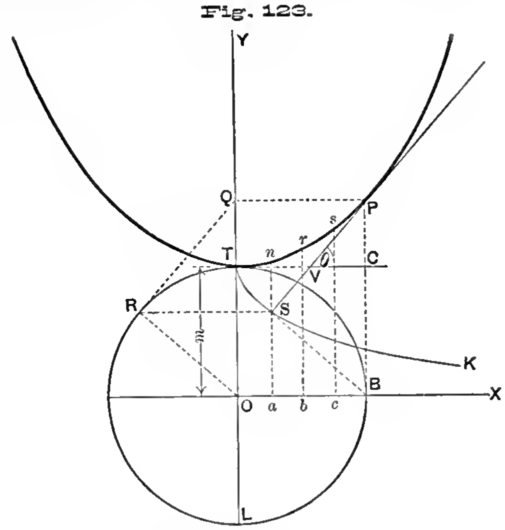
## THE CATENARY.

214. If a flexible chain, cable or string, of uniform weight per unit of length, be freely suspended by its extremities, the curve which it takes under the action of gravity is called a *Catenary*, from *catena, a chain.*

A simple and practical method of obtaining a catenary on the drawing-board would be to insert two pins in the board, in the desired relative position of the points of suspension, and then attach to them a string of the desired length. By holding the board vertically the string would assume the catenary, whose points could then be located with the pencil and joined in the usual manner with the irregular curve. Otherwise, if its points are to be located by means of an equation, we take axes in the plane of the curve, the $y$-axis (Fig. 123) being a vertical line through the lowest point $T$ of the catenary, while the $x$-axis is a horizontal line at a distance $m$ below $T$. The quantity $m$ is called the *parameter* of the curve, and is equal to the length of string which represents the tension at the lowest point.

The equation of the catenary[1] is then $y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)$ in which $e$ is the *base* of Napierian logarithms[2] and has the numerical value 2.7182818+.

Fig. 123.




By taking successive values of $x$ equal to $m$, $2m$, $3m$, etc., we get the following values for $y$:

$x = m \ldots y = \frac{m}{2}\left(e + \frac{1}{e}\right)$ which for $m = unity$ becomes 1.54308

$x = 2m \ldots y = \frac{m}{2}\left(e^2 + \frac{1}{e^2}\right)$ " " " " " 3.76217

$x = 3m \ldots y = \frac{m}{2}\left(e^3 + \frac{1}{e^3}\right)$ " " " " " 10.0676

$x = 4m \ldots y = \frac{m}{2}\left(e^4 + \frac{1}{e^4}\right)$ " " " " " 27.308

To construct the curve we therefore draw an arc of radius $OB = m$, giving $T$ on the axis of $y$ as the lowest point of the curve.

For $x = OB = m$ we have $y = BP = 1.54308$; for $x = Oa = \frac{m}{4}$ we have $y = an = 1.03142$.

The tension at any point $P$ is equal to the weight of a piece of rope of length $BP = PC + m$.

At the lowest point the tangent is horizontal. The length of any arc $TP$ is proportional to the angle $\theta$ between $TC$ and the tangent $PV$ at the upper extremity of the arc.

215. If a circle $RLB$ be drawn, of radius equal to $m$, it may be shown analytically that tangents $PS$ and $QR$, to catenary and circle respectively, from points at the same level, will be parallel: also that $PS$ equals the catenary-arc $PrT$; $S$ therefore traces the involute of the catenary, and as $SB$ always equals $RO$ and remains perpendicular to $PS$ (angle $ORQ$ being always 90°) we have the curve $TSK$ fulfilling the conditions of a *tractrix*. (See Art. 202.)

If a parabola, having a focal distance $m$, roll on a straight line, the focus will trace a catenary having $m$ for its parameter.

The catenary was mistaken by Galileo for a parabola. In 1669 Jungius proved it to be neither a parabola nor hyperbola, but it was not till 1691 that its exact mathematical nature was known, being then established by James Bernouilli.

## THE LOGARITHMIC OR EQUIANGULAR SPIRAL.

216. In Fig. 124 we have the curve called the *Logarithmic Spiral.* Its usual construction is based on the property that any radius vector, as $\rho$, which bisects the angle between two other radii, $OM$ and $ON$, is a mean proportional between them; i. e., $\rho^2 = OS^2 = OM \times ON$.

If $M$ and $G$ are points of the spiral we may find an intermediate point $K$ by drawing the ordinate $OK$ to a semicircle of diameter $OM + OG$; a perpendicular through $G$ to $GK$ will then give $D$, another point of the curve, and this construction may be repeated indefinitely.

Radii making equal angles with each other are evidently in geometrical progression.

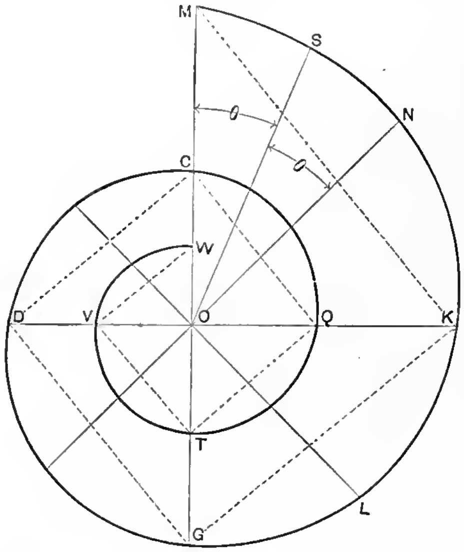
This spiral is often called *Equiangular* from the fact that the angle is always the same between

[1] Rankine. *Applied Mechanics.* Art. 175.

[2] In the expression $10^2 = 100$ the quantity "2" is called the *logarithm* of 100, it being the exponent of the power to which 10 must be raised to give 100. Similarly 2 would be the logarithm of 64, were 8 the *base* or number to be raised to the power indicated.

a radius vector and the tangent at its extremity. Upon this property is based its use as the outline for spiral cams and for lobed wheels. The curve never reaches the pole.

Fig. 124.




The name *logarithmic spiral* is based on the property that the angle of revolution is proportional to the logarithm of the radius vector. This is expressed by $\rho = a^{\theta}$, in which $\theta$ is the varying angle, and $a$ is some arbitrary constant.

*To construct a tangent* by calculation, divide the hyperbolic logarithm[1] of the ratio $OM:OK$ (which are any two radii whose values are known) by the angle between these radii, expressed in circular measure;[2] the quotient will be the tangent of the constant angle of obliquity of the spiral.

217. Among the more interesting properties of this curve are the following:

Its involute is an equal logarithmic spiral.

Were a light placed at the pole, the caustic—whether by reflection or refraction—would be a logarithmic spiral.

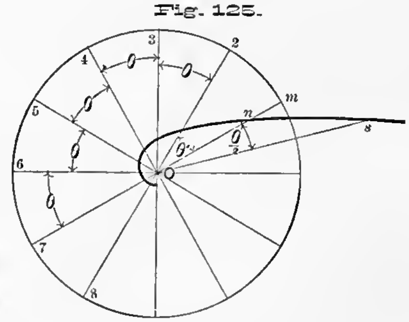
The discovery of these properties of recurrence led James Bernouilli to direct that this spiral be engraved on his tomb, with the inscription—*Eadem Mutata Resurgo*, which, freely translated, is—*I shall arise the same, though changed.*

Kepler discovered that the orbits of the planets and comets were conic sections having a focus at the centre of the sun. Newton proved that they would have described logarithmic spirals as they travelled out into space, had the attraction of gravitation been inversely as the *cube* instead of the *square* of the distance.

### THE HYPERBOLIC OR RECIPROCAL SPIRAL.

218. In this spiral the length of a radius vector is in inverse ratio to the angle through which it turns. Like the logarithmic spiral, it has an infinite number of convolutions about the pole, which it never reaches.

Fig. 125.




The invention of this curve is attributed to James Bernouilli, who showed that Newton's conclusions as to the logarithmic spiral (see Art. 217) would also hold for the hyperbolic spiral, the initial velocity of projection determining which trajectory was described.

To obtain points of the curve divide a circle $m\,5\,8$ (Fig. 125) into any number of equal parts, and on some initial radius $O\,m$ lay off some unit, as an inch; on the second radius $O\,2$ take $\frac{O\,n}{2}$; on the third $\frac{O\,n}{3}$, etc. For one-half the angle $\theta$ the radius vector would evidently be $2\,O\,n$, giving a point $s$ outside the circle.

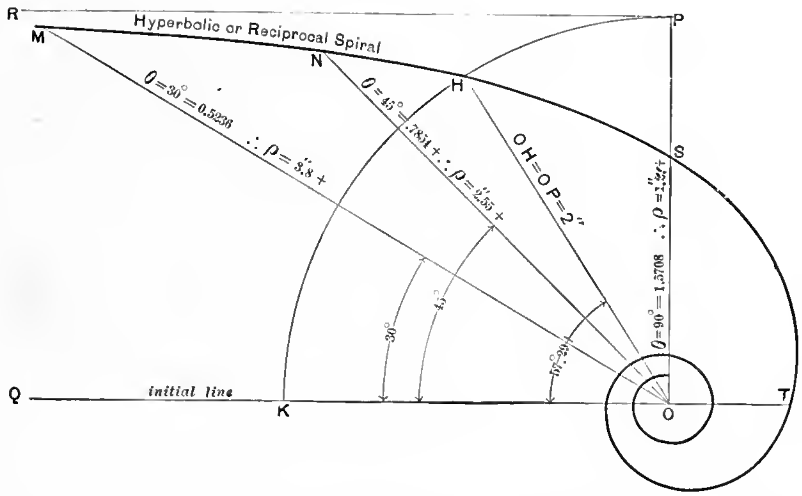
The equation to the curve is $\frac{1}{r} = a\,\theta$, in which $r$ is the radius vector, $a$ some numerical constant, and $\theta$ is the angular rotation of $r$ (in circular measure) estimated from some initial line.

[1] To get the hyperbolic logarithm of a number multiply its common logarithm by 2.3026.

[2] In circular measure $360° = 2\pi r$, which, for $r = 1$, becomes 6.28318; $180° = 3.14159$; $90° = 1.5708$; $60° = 1.0472$; $45° = 0.7854$; $30° = 0.5236$; $1° = 0.0174533$.

The curve has an asymptote parallel to the initial line, and at a distance from it equal to $\frac{1}{a}$ units.

Fig. 126.




To construct the spiral from its equation take $O$ as the pole (Fig. 126); $OQ$ as the initial line; $a$, for convenience, some fraction, as $\frac{1}{4}$; and as our *unit* some quantity, say half an inch, that will make $\frac{1}{a}$ of convenient size. Then, taking $QO$ as the initial line, make $OP = \frac{1}{a} = 2''$, and draw $PR$ parallel to $OQ$ for the asymptote. For $\theta = 1$, that is, for *arc* $KH =$ *radius* $OH$, we have $r = \frac{1}{a} = 2''$, giving $H$ for one point of the spiral. Writing the equation in the form $r = \frac{1}{a} \cdot \frac{1}{\theta}$ and expressing various values of $\theta$ in circular measure we get the following:

$\theta = 30° = 0.5236$; $r = OM = 3''.8 +$: $\theta = 45° = 0.7854$; $r = ON = 2''.55$;

$\theta = 90° = 1.2708$; $r = OS = 1''.2 +$: $\theta = 180° = 3.14159$; $r = OT = .6366$, etc.

*The tangent to the curve* at any point makes with the radius vector an angle $\phi$ which is found by analysis to sustain to the angle $\theta$ the following trigonometrical relation, *tan* $\phi = \theta$; the circular measure of $\theta$ may therefore be found in a table of natural tangents, and the corresponding value of $\phi$ obtained.

### THE LITUUS.—THE IONIC VOLUTE.

219. *The Lituus* is a spiral in which the radius vector is inversely proportional to the square root of the angle through which it has revolved. This relation is shown by the equation $r = \frac{1}{a\sqrt{\theta}}$, also written $a^2\theta = \frac{1}{r^2}$.

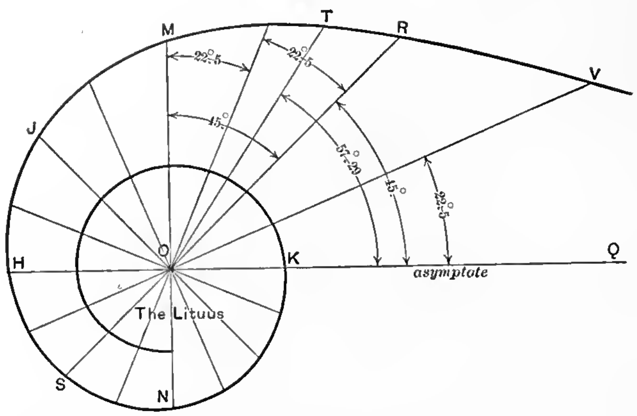
When $\theta = 0$ we find $r = \infty$, which makes the initial line an asymptote to the curve.

In Fig. 127 take $OQ$ as the initial line, $O$ as the pole, $a = 2$, and as our unit $3''$; then $\frac{1}{a} = 1\frac{1}{2}''$.

For $\theta = 90° = \pi$ (in circular measure 1.5708) we have $r = OM = 1''.2 +$. For $\theta = 1$ we have the radius $OT$ making an angle of $57°.29 +$ with the initial line, and in length equal to $\frac{1}{a}$ units,

i. e., $1\frac{1}{2}''$. For $\theta = 45° = \frac{\pi}{4}$ (or 0.7854) $r$ will be $OR = 1''.7 +$. Then $OH = \frac{OR}{2}$; for in rotating to $OH$ the radius vector passes over four 45° angles, and the radius must therefore be one-half what it was for the first 45° described. Similarly, $OK = \frac{OM}{2}$; $OM = \frac{OV}{2}$, etc.; this relation enabling the student to locate any number of points.

Fig. 127.

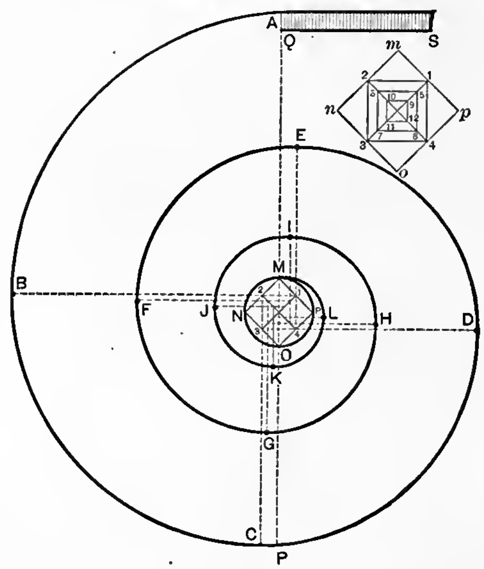



*To draw a tangent to the curve* we employ the relation $\tan \phi = 2\,\theta$, $\phi$ being the angle made by the tangent line with the radius vector, while $\theta$ is the angular rotation of the latter, in circular measure.

*Architectural Scrolls.—The Ionic Volute.* The Lituus and other spirals are occasionally employed as volutes and other architectural ornaments. In the former application it is customary for the spiral to terminate on a circle called the *eye*, into which it blends tangentially.

Usually, in practice, circular-arc approximations to true spiral forms are employed, the simplest of which, for the scroll on the capital of an Ionic column, is probably the following:

Fig. 127 (a).




Taking $AOP$, the total height of the volute, at sixteen of the eighteen "parts" into which the *module* (the unit of proportion = the semi-diameter of the column) is divided, draw the circular eye with *radius* equal to *one* such part, the centre dividing $AP$ into segments of *seven* and *nine* parts respectively. Next inscribe in the eye a square with one diagonal vertical; parallel to its sides draw (see enlarged square $mnop$) 2—4 and 3—1, and divide each into six equal parts, which number up to twelve, as indicated. Then (returning to main figure) the arc $AB$ has centre 1 and radius 1—$A$. With 2 as a centre draw arc $BC$; then $CD$ from centre 3, etc.

In the complete drawing of an Ionic column the centre of the eye would be at the intersection of a vertical line from the lower extremity of the cyma reversa with a horizontal through the lower line of the echinus. To complete the scroll a second spiral would be required, constructed according to the same law and beginning at $Q$, where $AQ$ is equal to one-half part of the module.

# *CHAPTER VI.*

## TINTING—FLAT AND GRADUATED.—MASONRY, TILING, WOOD GRAINING, RIVER-BEDS AND OTHER SECTIONS, WITH BRUSH ALONE OR IN COMBINED BRUSH AND LINE WORK.

220. Brush-work, with ink or colors, is either *flat* or *graduated.* The former gives the effect of a flat surface parallel to the paper on which the drawing is made, while graded tints either show curvature, or—if indicating flat surfaces—represent them as inclined to the paper, i. e., to the plane of projection. For either, the paper should be, as previously stated (Arts. 41 and 44) *cold-pressed* and *stretched.*

The surface to be tinted should not be abraded by sponge, knife or rubber.

221. The liquid employed for tinting must be free from sediment; or at least the latter, if present, must be allowed to settle, and the brush dipped only in the clear portion at the top. Tints may, therefore, best be mixed in an artist's water-glass, rather than in anything shallower. In case of several colors mixed together, however, it would be necessary to thoroughly stir up the tint each time before taking a brushful.

A tint prepared from a *cake* of high-grade India ink is far superior to any that can be made by using the ready-made liquid drawing inks.

222. The size of brush should bear some relation to that of the surface to be tinted; large brushes for large surfaces and vice versa. The customary error of beginners is to use too small and too dry a brush for tinting, and the reverse for shading.

223. Harsh outlines are to be avoided in brush work, especially in handsomely shaded drawings, in which, if sharply defined, they would detract from the general effect. This will become evident on comparing the spheres in Figs. 1 and 4 of Plate II.

Since tinting and shading can be successfully done, after a little practice, with only *pencilled* limits, there is but little excuse for inking the boundaries; but if, for the sake of definiteness, the outlines are inked at all it should be *before* the tinting, and in the finest of lines, preferably of "water-proof" ink; although any ink will do provided a soft sponge and plenty of clean water be applied to remove any excess that will "run." The sponge is also to be the main reliance of the draughtsman for the correction of errors in brush work; the water, however, and not the friction to be the active agent. An entire tint may be removed in this way in case it seems desirable.

224. When beginning work incline the board at a small angle, so that the tint will flow down after the brush. For a *flat,* that is, a *uniform* tint, start at the upper outline of the surface to be covered, and with a brush full, yet not surcharged—which would prevent its coming to a good point—pass lightly along from left to right, and on the return carry the tint down a little farther, making short, quick strokes, with the brush held almost perpendicular to the paper. Advance the tint as evenly as possible along a horizontal line; work quickly *between* outlines, but more slowly *along* outlines, as one should never overrun the latter and then resort to "trimming" to conceal lack of skill. It is possible for any one, with care and practice, to tint *to* yet not *over* boundaries.

*The advancing edge of the tint must not be allowed to dry until the lower boundary is reached.*

No portion of the paper, however small, should be missed as the tint advances, as the work is likely to be spoiled by retouching.

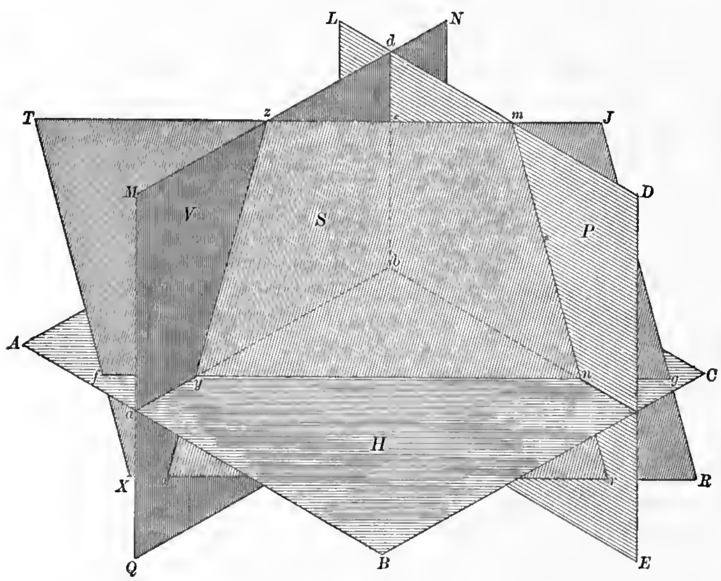
Should any excess of tint be found along the lower edge of the figure it should be absorbed by the brush, after first removing the latter's surplus by means of blotting paper.

To get a dark effect several medium tints laid on in succession, each one drying before the next is applied, give better results than one dark one.

The heightened effect described in Art. 72, viz., a line of light on the upper and left-hand edges, may be obtained either (a) by ruling a broad line of *tint* with the drawing-pen at the desired distance from the outline, and instantly, before it dries, tinting from it with the brush; or (b) by ruling the line with the pen and thick Chinese White.

225. A tint will spread much more evenly on a large surface if the paper be first slightly dampened with clean water. As the tint will follow the water, the latter should be limited exactly to the intended outlines of the final tint.

Fig. 128.




226. Of the colors frequently used by engineers and architects those which work best for flat effects are carmine, Prussian blue, burnt sienna and Payne's gray. Sepia and Gamboge, are, fortunately, rarely required for uniform tints; but the former works ideally for *shading* by the "dry" process described in the next article; and its rich brown gives effects unapproachable with anything else. It has, however, this peculiarity, that repeated touches upon a spot to make it darker produce the opposite effect, unless enough time elapses between the strokes to allow each addition to dry thoroughly.

227. For elementary practice with the brush the student should lay flat washes, in India tints, on from six to ten rectangles, of sizes between $2'' \times 6''$ and $6'' \times 10''$. If successful with these his next work may be the reproduction of Fig. 128, in which *H*, *V*, *P* and *S* denote horizontal, vertical, profile and section planes respectively. The figure should be considerably enlarged.

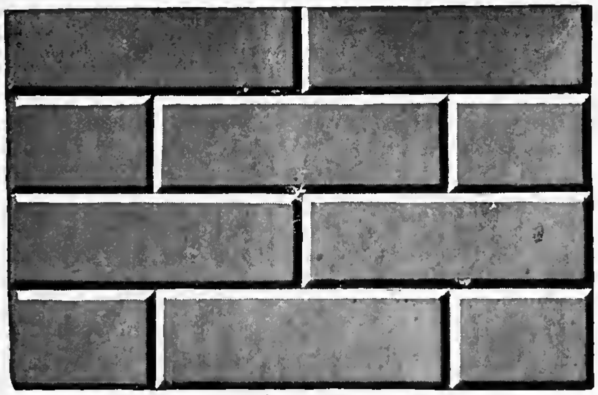
The plane *V* may have two washes of India ink; *H* one of Prussian blue; *P* one of burnt sienna, and *S* one of carmine.

The edges of the planes *H*, *V* and *P* are either vertical or inclined 30° to the horizontal.

For the section-plane assume $n$ and $m$ at pleasure, giving direction $nm$, to which $JR$ and $TX$ are parallel. A horizontal, $mz$, through $m$ gives $z$. From $n$ a horizontal, $ny$, gives $y$ on $ab$. Joining $y$ with $z$ gives the "trace" of $S$ on $V$.

228. Figures 129 and 130 illustrate the use of the brush in the representation of masonry. The former may be altogether in ink tints, or in medium burnt umber for the front rectangle of

Fig. 129.

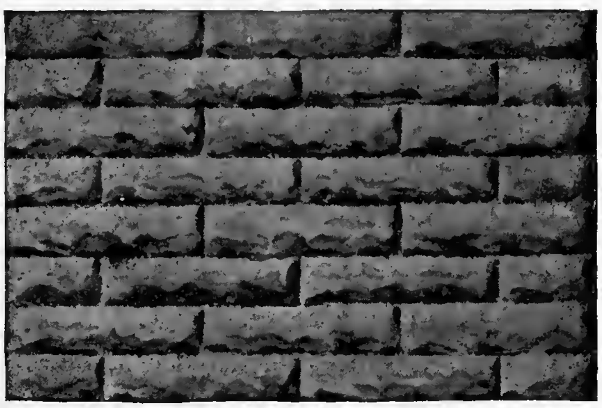


each stone, and dark tint of the same, directly from the cake, for the bevel. Lightly pencilled limits of bevel and rectangle will be needed; no inked outlines required or desirable.

The last remark applies also to Fig. 130, in which "quarry-faced" ashlar masonry is represented. If properly done, in either burnt umber or sepia, this gives a result of great beauty, especially effective on the piers of a large drawing of a bridge.

The darker portions are tinted directly from the cake, and are purposely made irregular and "jagged" to reproduce as closely as possible the fractured appearance of the stone.

Fig. 130.

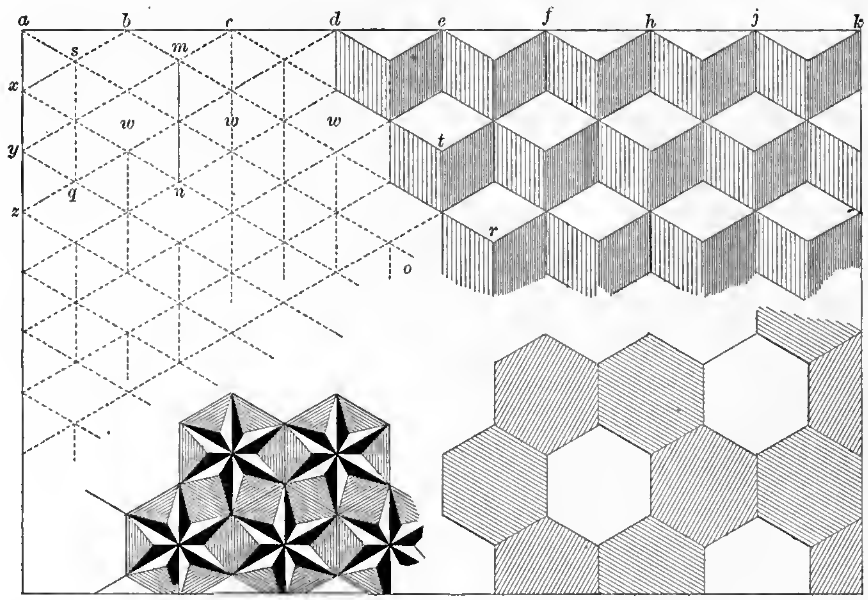


Two brushes are required when an "over-hang" or jutting portion is to be represented, one with a medium tint, the other with the thick color, as before. An irregular line being made with the latter, the tint is then softened out *on the lower side* with the point of the brush having the lighter tint. A light wash of the intended tone of the whole mass is quickly laid over each stone, either *before* or *after* the irregularities are represented, according as an exceedingly angular or a somewhat softened and rounded effect is desired.

229. Designs in tiling are excellent exercises, not only for brush work in flat tints, but also—in their preliminary construction—in precision of line work. The superbly illustrated catalogues of the Minton Tile Works are, unfortunately, not accessible by all students, illustrating as they do, the finest and most varied work in this line, both of designer and chromo-lithographer; but it is quite within the bounds of possibility for the careful draughtsman to closely approach if not equal the standard and general appearance of their work, and as suggestions therefor Figs. 131 and 132 are presented.

230. In Fig. 131 the upper boundary, $adhk$, of a rectangle is divided at $a$, $b$, $c$, etc., into equal spaces, and through each point of division two lines are drawn with the 30° triangle, as $bx$ and $br$ through $b$. The oblique lines terminate on the sides and lower line of the rectangle. If the work is accurate—and it is worthless if not—any vertical line as $mn$, drawn through the intersection, $m$, of a pair of oblique lines, will pass through the intersection of a series of such pairs.

Fig. 131.



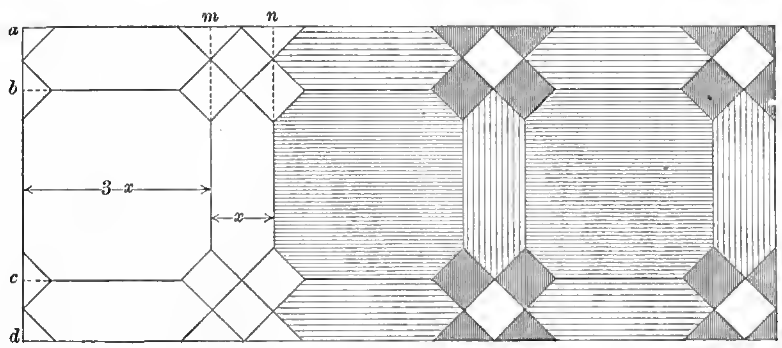

The figure shows three of the possible designs whose construction is based on the dotted lines of the figure. For that at the top and right, in which *horizontal* rows of rhombi are left white, we draw vertical lines as $sq$ and $mn$ from the lower vertex of each intended white rhombus, continuing it over two rhombi, when another white one will be reached. The dark faces of the design are to be finally in solid black, previous to which the lighter faces should be tinted with some drab or brown tint. The pencilled construction lines would necessarily be erased before the tint was laid on.

The most opaque effect in colors is obtained by mixing a large portion of Chinese white with the water color, making what is called by artists a "body color." Such a mixture gives a result in marked contrast with the transparent effect of the usual wash; but the amount of white used should be sufficient to make the tint in reality a *paste*, and no more should be taken on the brush at one time than is needed to cover one figure.

Sepia and Chinese white, mixed in the proper proportions, give a tint which contrasts most agreeably with the black and white of the remainder of the figure. The star design and the hexagons in the lower right-hand corner result from extensions or modifications of the construction just described which will become evident on careful inspection.

231. Fig. 132 is a Minton design with which many are familiar, and which affords opportunity for considerable variety in finish. Its construction is almost self-evident. The *equal* spaces, $a\,b$, $c\,d$, $m\,n$—which may be any width, $x$,—alternate with other equal spaces $b\,c$, which may preferably be about $3\,x$ in width. Lines at 45°, as indicated, complete the preliminaries to tinting.

Fig. 132.




The octagons may be in Prussian blue, the hexagons in carmine, and the remainder in white and black, as shown; or browns and drabs may be employed for more subdued effects.

## SHADING.

232. For *shading*, by graduated tints, provide a glass of clear water in addition to the tint; also an ample supply of blotting paper.

The water-color or ink tint may be considerably darker than for flat tinting; in fact, the darker it is, provided it is clear, the more rapidly can the desired effect be obtained.

The brush must contain much less liquid than for flat work.

Lay a narrow band of tint quickly along the part that is to be the darkest, then dip the brush into clear water and immediately apply it to the blotter, both to bring it to a good point and to remove the surplus tint. With the now once-diluted tint carry the advancing edge of the band slightly farther. Repeat the operation until the tint is no longer discernible as such.

The process may be repeated from the same starting point as many times as necessary to produce the desired effect; but the work should be allowed to dry each time before laying on a new tint.

Any irregularities or streaks can easily be removed after the work dries, by retouching or "stippling" with the point of a fine brush that contains *but little tint*—scarcely more than enough to enable the brush to retain its point. For small work, as the shading of rivets, rods, etc., the process just mentioned, which is also called "dry shading," is especially adapted, and, although somewhat tedious, gives the handsomest effects possible to the draughtsman.

233. Where a good, *general* effect is wanted, to be obtained in less time than would be required for the preceding processes, the method of over-lapping flat tints may be adopted. A narrower band of dark tint is first laid over the part to be the darkest. When dry this is overlaid by a broader band of lighter tint. A yet lighter wash follows, beginning on the dark portion and extending still farther than its predecessor. The process is repeated with further diluted tints until the desired effect is obtained.

Faintly-pencilled lines may be drawn at the outset as limits for the edges of the tints.

This method is better adapted for large work, that is not to be closely scrutinized, than for drawings that deserve a high degree of finish.

234. As to the relative position and gradation of the lights and shades on a figure, the student is referred to Arts. 78 and 79 and the chapter on shadows; also to the figures of Plate II, which may serve as examples to be imitated while the learner is acquiring facility in the use of the brush, and before entering upon constructive work in shades and shadows. Fig. 3 of Plate II may be undertaken first, and the contrast made yet greater between the upper and lower boundaries. Fig. 1 (Plate II) requires no explanation. In Fig. 133 we have a wood-cut of a sphere, with the theoretical dark or "shade" line more sharply defined than in the spheres on the plate.

Fig. 133.

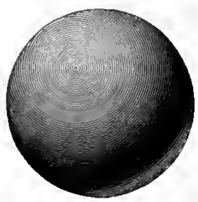
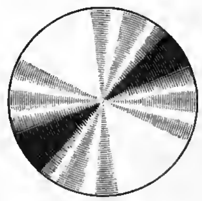


Fig. 134.



A drawing of the end of a highly-polished revolving shaft, or even of an ordinary metallic disc, would be shaded as in Fig. 134.

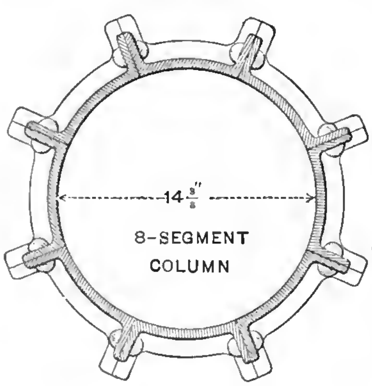
Fig. 2 (Plate II) represents the triangular-threaded screw, its oblique surfaces being, in mathematical language, *warped helicoids*, generated by a moving straight line, one end of which travels *along the axis* of a cylinder while the other end traces or follows a *helix* on the cylinder.

The construction of the helix having already been given (Art. 120) the outlines can readily be drawn. The method of exactly locating the shadow and shade lines will be found in the chapter on shadows.

Fig. 4 (Plate II), when compared with Fig. 91, illustrates the possibilities as to the representation of interesting mathematical relations. The fact may again be mentioned, on the principle of "line upon line," as also for the benefit of any who may not have read all that has preceded, that the spheres in the cone are tangent to the oblique plane at the *foci* of the elliptical section. The peculiar dotted effect in this figure is due to the fact that the original drawing, of which this is a photographic reproduction by the gelatine process, was made with a lithographic crayon upon a special pebbled paper much used by lithographers. The original of Fig. 1, on the other hand, was a brush-shaded sphere on Whatman's paper.

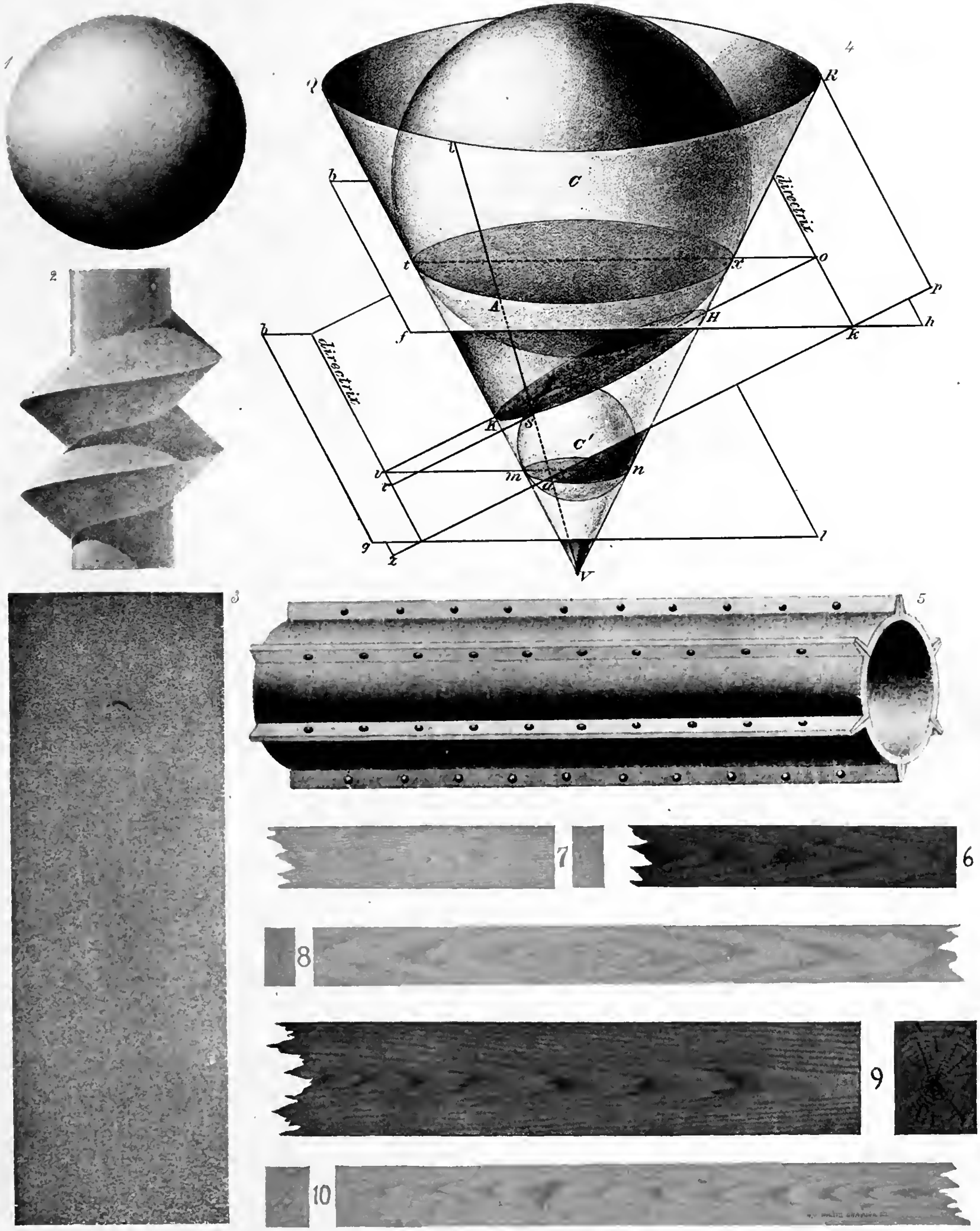
235. Fig. 5 (Plate II) shows a "Phœnix column," the strongest form of iron for a given weight, for sustaining compression. The student is familiar with it as an element of outdoor construction in bridges, elevated railroads, etc.; also in indoor work in many of the higher office buildings of our great cities.

Fig. 135.



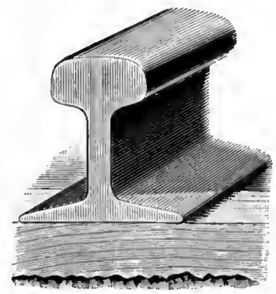

By drawing first an end view of a Phœnix column, similar to that of Fig. 135, we can readily derive an oblique view like that of the plate, by including it between parallels from all points of the former. The proportions of the columns are obtainable from the tables of the company.

Fig. 135 is a cross-section of the 8-segment column, the shaded portion showing the minimum and the other lines the maximum size for the same inside diameter.


1
2
3
4
5
6
7
8
9
10
directrix
directrix
C
C′
V

In a later chapter the proportions of other forms of structural iron will be found. Short lengths of any of these, if shown in oblique view, are good subjects for the brush, especially for "dry" shading, the effect to be aimed at being that of the rail section of Fig. 136.

Fig. 136.



236. When some particular material is to be indicated, a flat tint of the proper technical color (see Art. 73) should be laid on with the brush, either before or after shading. When the latter is done with sepia it is probably safer to lay on the flat tint first.

A darker tint of the technical color should always be given to a cross-section. For blue-printing, a cross-section may be indicated in solid black.

WOOD.—RIVER-BEDS.—MASONRY, ETC.

237. While the engineering draughtsman is ordinarily so pressed for time as not to be able to give his work the highest finish, yet he ought to be able, when occasion demands, to obtain both natural and artistic effects; and to conduce to that end the writer has taken pains to illustrate a number of ways of representing the materials of construction. Although nearly all of them may be—and in the cuts are—represented in black and white (with the exception of the wood-graining on Plate II), yet colors, in combined brush and line work, are preferable. The student will, however, need considerable practice with pen and ink before it will be worth while to work on a tinted figure.

238. Ordinarily, in representing wood, the mere fact that it *is* wood is all that is intended to be indicated. This may be done most simply by a series of irregular, approximately-parallel lines, as in Fig. 10 or as on the rule in Fig. 17, page 12. Make no attempt, however, to have the grain *very* irregular. The natural unsteadiness of the hand, in drawing a long line toward one continuously, will cause almost all the irregularity desired.

If a better effect is wanted, yet without color, the lines may be as in Fig. 107, which represents hard wood.

In graining, the draughtsman should make his lines *toward* himself, standing, so to speak, at the end of the plank upon which he is working.

The splintered end of a plank should be sharply toothed, in contradistinction to a metal or stone fracture, which is what might be called smoothly irregular.

239. An examination of any piece of wood on which the grain is at all marked will show that it is darker at the inner vertex of any marking than at the outer point. Although this difference is more readily produced with the brush, yet it may be shown in a satisfactory degree with the pen, by a series of after-touches.

240. If we fill the pen with a rather dark tint of the conventional color, draw the grain as in the figures just referred to, and then overlay all with a medium flat wash of some properly chosen color, we get effects similar to those of Plate II.

On large timber-work the preliminary graining, as also the final wash, may be done altogether with the brush; as was the original of Fig. 9, Plate II.

End views of timbers and planks are conventionally represented by a series of concentric free-hand rings in which the spacing increases with the distance from the heart; these are overlaid with a few radial strokes of darker tint. In ink alone the appearance is shown in Figs. 39 and 115.

241. The color-mixtures recommended by different writers on wood graining are something short of infinite in number; but with the addition of one or two colors to those listed in the draughtsman's outfit (Art. 56) one should be able to imitate nature's tints very closely.

No hard-and-fast rule as to the proportions of the colors can be given. In this connection we may quote Sir Joshua Reynolds' reply to the one who inquired how he mixed his paints. "With brains," said he. One *general* rule, however; always employ delicate rather than glaring tints.

Merely to indicate wood with a color and no graining use burnt sienna, the tint of Figs 7, 8 and 10 of Plate II.

Drawing from the writer's experience and from the suggestions of various experimenters in this line the following hints are presented:—

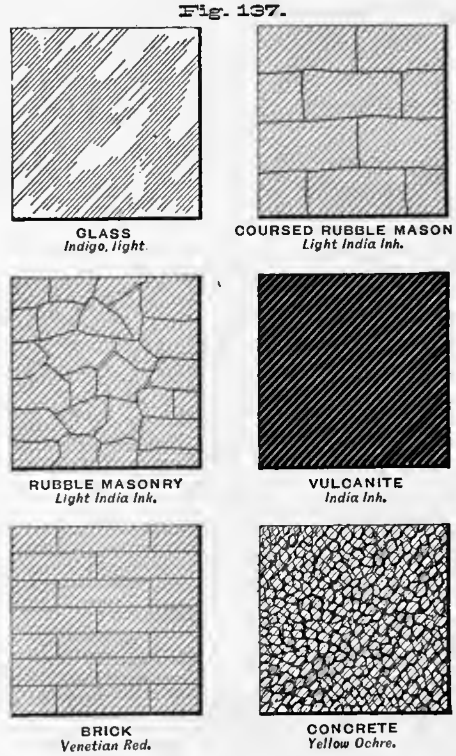
*In every case grain first*, then overlay with the ground tint, which should always be much lighter than the color used for the grain. If possible have at hand a good specimen of the wood to be imitated.

*Hard Pine:* Grain—burnt umber with either carmine or crimson lake; for overlay add a little gamboge to the grain-tint diluted.

*Soft Pine:* Gamboge or yellow ochre with a small amount of burnt sienna.

*Black Walnut:* Grain—burnt umber and a very little dragon's blood; final overlay of modified tint of the same or with the addition of Payne's gray.

Fig. 137.




*Oak:* Grain—burnt sienna; for overlay, the same, with yellow ochre.

*Chestnut:* Grain—burnt umber and dragon's blood; overlay of the same, diluted, and with a large proportion of gamboge or light yellow added.

*Spruce:* Grain—burnt umber, medium; add yellow ochre for the overlay.

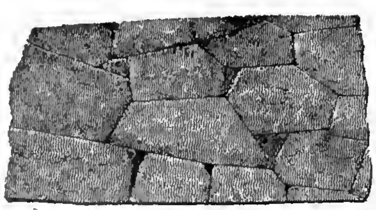
*Mahogany:* Grain—burnt sienna or umber with a small amount of dragon's blood; dilute, and add light yellow for the overlay.

*Rosewood:* Grain—replace the dragon's blood of mahogany-grain by carmine, and for overlay dilute and add a little Prussian blue.

242. *River-beds* in black and white or in colors have been already treated in Art. 26, to which it is only necessary to add that such sections are usually made quite narrow, and, preferably—if in color—shaded quite abruptly on the side opposite the water.

243. The sections of *masonry*, *concrete*, *brick*, *glass* and *vulcanite*, given on page 25 as pen and ink exercises, are again presented in Fig. 137, for reproduction in combined brush and line work. The appropriate color is indicated under each section.

Fig. 138.

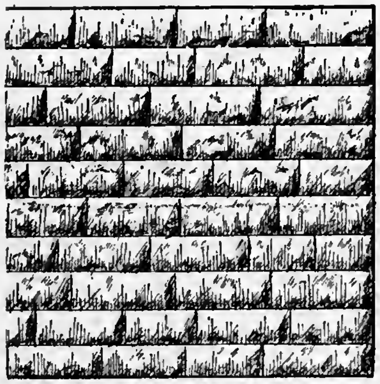


244. *Masonry* constructions may be broadly divided into *rubble* and *ashlar*.

In *ashlar* masonry the bed-surfaces and the joints (edges) are shaped and dressed with great care, so that the stones may not only be placed in regular layers or *courses*, but often fill exactly some predetermined place, as in arch construction, in which case the determination of their forms and the derivation of the patterns for the stone-cutter involves the application of the Descriptive Geometry of Monge. (Art. 283).

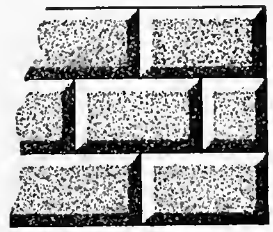
*Rubble* work, however, consists of constructions involving stones mainly "in the rough," but may be either coursed or uncoursed. Fig. 138 is a neat example of uncoursed though partially dressed or "hammered" rubble. In section, as shown in Fig. 137, it is merely necessary to rule section-lines over the boundaries of the stones—a remark applying equally to ashlar masonry.

Fig. 139.



The other examples in this chapter are of ashlar, mainly "quarry-faced," that is, with the front nearly as rough as when quarried. A beveled or "chamfered" ashlar is shown in Figs. 129 and 140, the latter shaded in what is probably the most effective way for small work, viz., with *dots*, the effect depending upon the *number*, not the *size* of the latter.

Fig. 140.



Only a careful examination of the *kind* and *position* of the lines in the other figures on this page will disclose the secret of the variety in the effects produced. For the handsomest results with any of these figures the pen-work—

Fig. 141.

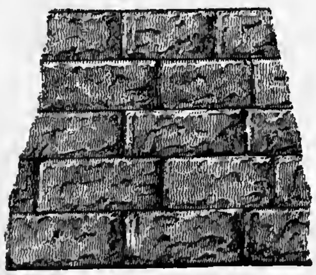
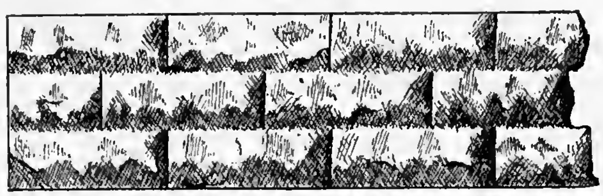


Fig. 142.



whether dotting or "cross-hatching"—should be preceded by an undertone of either India ink, umber, Payne's gray, cobalt or Prussian blue, according to the kind of stone to be represented.

Fig. 143.

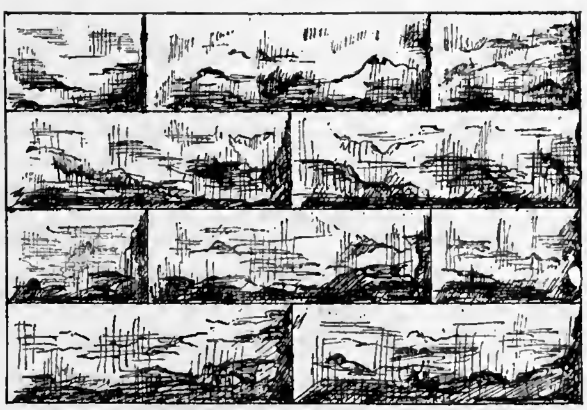
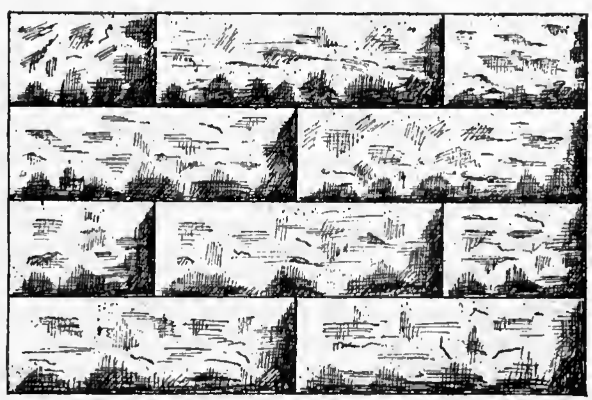


Fig. 144.



For slate use a pale blue; for brown free-stone either an umber or sepia; while for stone in general, *kind* immaterial, use India ink.

## CHAPTER VII.

### FREE-HAND AND MECHANICAL LETTERING.—PROPORTIONING OF TITLES.

245. Practice in lettering forms an essential part of the elementary work of a draughtsman. Every drawing has to have its title, and the general effect of the result as a whole depends largely upon the quality of the lettering.

Other things being equal, the expert and rapid draughtsman in this line has a great advantage over one who can do it but slowly. For this reason free-hand lettering is at a high premium, and the beginner should, therefore, aim not only to have his letters *correctly formed* and *properly spaced*, but, as far as possible, to do without mechanical aids in their construction. When under great pressure as to time it is, however, perfectly legitimate to employ some of the mechanical expedients used in large establishments as "short cuts" and labor-savers. Among these the principal are "tracing" and the use of rubber types.

246. To *trace* a title one must have at hand complete printed alphabets *of the size of type required.* Placing a piece of tracing-paper over the letter wanted, it is traced with a hard pencil, the paper then slipped along to the next letter needed, and the process repeated until the words desired have been outlined. The title is then transferred to the drawing by first running over the lines on the back of the tracing-paper with a soft pencil, after which it is only necessary to re-trace the letters with a hard pencil, on the face of the transfer-paper, to find their outlines faintly yet sufficiently indicated on the paper underneath. Carbon paper may also be used for transferring.

247. The process just described would be of little service to a ready free-hand draughtsman, but with the use of *rubber types*, for the words most frequently recurring in the titles, a merely average worker may easily get results which—in point of time—cannot be exceeded by any other method. When employing such types either of the following ways may be adopted: (a) a light impression may be made with the aniline ink ordinarily used on the pads, and the outlines then followed and the "filling in" done either with a writing-pen* or fine-pointed sable-hair brush; or (b) the impression may be made after moistening the types on a pad that has been thoroughly wet with a light tint of India ink. The drawing-ink must then be immediately applied, free-hand, with a Falcon pen or sable brush, before the type-impression can dry. The pen need only be passed down the middle of a line, as on the dampened surface the ink will spread instantly to the outlines.

248. The educated draughtsman should, however, be able not only to draw a legible title of the simple character required for shop-work, and in which the foregoing expedients would be mainly serviceable, but be prepared also for work out of the ordinary line, and, if need be, quite elaborate, as on a competitive drawing. Such knowledge can only be gained by careful observation of the forms of letters, and considerable practice in their construction.

No rigid rules can be laid down as to choice of alphabets for the various possible cases. Common-sense, custom and a natural regard for the "fitness of things" are the determining factors.

Obviously rustic letters would be out of place on a geometrical drawing, and other incongruities

---

*Refer to Art. 27 with regard to the pens to be used for the various styles of letters

will naturally suggest themselves. In addition to the hints in Art. 27 a few general principles and methods may, however, be stated to the advantage of the beginner, who should also refer to the special instructions given in connection with certain specimen alphabets at the end of this work.

249. In the first place, a title should be symmetrical with respect to a vertical centre-line, a rule which should be violated but rarely, and then, usually, when the title is to be somewhat fancy in design, as for a magazine cover.

Elementary Plates
in
MECHANICAL DRAWING
drawn by Cortlandt Van Corlear at the
LEADING TECHNICAL SCHOOL
Jan.-June, 3001.

250. If it be a *complete* as distinguished from a *partial* or *sub*-title it will answer the following questions which would naturally arise in the mind of the examiner:—

What is it?—Where done?—By whom?—When?—On what scale?

In answering these questions the relative valuation and importance of the lines are expressed by the *sizes* and *kinds* of type chosen. This is a point requiring most careful consideration, as the final effect depends largely upon a proper balancing of values.

DETAIL DRAWINGS
OF
PERFECTION SUSPENSION BRIDGE
designed by
Goodwin, Mackenzie and Cartwright
MINNEAPOLIS, MINN.
SCALE 4 FT. = 1 IN. June 16, 2900. JOSE MARTINEZ, DEL.

251. The "By whom?" may cover two possibilities. In the case of a set of drawings made in a scientific school it would refer to the *draughtsman*, and his name might properly have considerably greater prominence than in any other case. The upper title on this page is illustrative of this point, as also of a symmetrical and balanced arrangement, although cramped as to space, vertically.

Ordinarily the "By whom?" will refer to the designer, and the draughtsman's name ought to be comparatively inconspicuous, while the name of the designer should be given a fair degree of prominence. This, and other important points to be mentioned, are illustrated in the preceding

arrangement, printed, like the upper title, from types of which complete alphabets will be found at the end of this work.

252. The abbreviation *Del.*, often placed after the draughtsman's name, is for *Delineavit—He drew it*—and does not indicate what the visitor at the exhibition supposed, that all good draughtsmen hail from Delaware.

253. The best designed titles are either in the form of two truncated pyramids having, if possible, the most important line as their common base, or else elliptical in shape.

254. The use of capitals throughout a line depends upon the style of type. It gives a most unsatisfactory result if the letters are of irregular outline, as is amply evidenced by the words

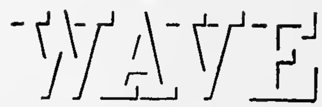
MECHANICAL DRAWING,

each letter of which is exquisite in form, but the combination almost illegible. Contrast them with the same style, but in capitals and small letters:—

Mechanical Drawing.

255. As to *spacing*, the visible white spaces between the letters should be as nearly the same as possible. In this feature, as in others, the draughtsman can get much more pleasing results than the printer, since the latter usually has each letter on a separate piece of metal, and can not adjust his space to any particular combination of letters, such as F A, L V, W A or A V, where a better effect would be obtained by placing the lower part of one letter under the upper part of the next. This is illustrated in Fig. 146, which may be contrasted with the printer's best spacing of the separate types for the A and W in the word "Drawings" of the last title.

Fig. 146.



256. The *amount of space* between letters will depend upon the length of line that the word or words must make. If an important word has few letters they should be "spaced out," and the letters themselves of the "extended" kind, i. e., broader than their height. The following word will illustrate. The characteristic feature of this type, viz., heavy horizontals and light verticals, is common to all the variations of a fundamental form frequently called *Italian Print.*

B R I D G E.

When, on the other hand, many letters must be crowded into a small space, a "condensed" style of letter must be adopted, of which the following is an example:

**Pennsylvania Railroad.**

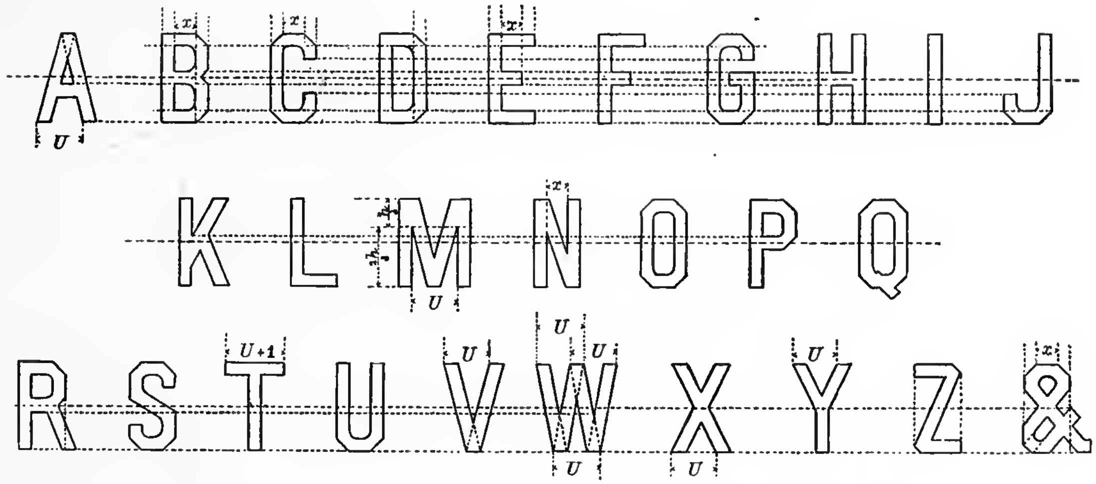
257. While the varieties of letters are very numerous yet they are all but changes rung on a few fundamental or basal forms, the most elementary of which is the

## GOTHIC, ALSO CALLED HALF-BLOCK.

Letters like B, O, etc., which have, usually, either few straight parts or none at all, may, for the sake of variety as also for convenience of construction, be made partially or wholly angular; in the latter case the form is called *Geometric Gothic* by some type manufacturers. It is only appropriate for work exclusively mechanical. The rounded forms are preferable for free-hand lettering.

The following complete Gothic alphabet is so constructed that whether designed in its "condensed" or "extended" form the proper proportions may be easily preserved.

Fig. 147.



Taking all the solid parts of the letters at the same width as the I, we will find any letter *of average width*, as U, to be twice that unit, plus the opening between the uprights, which last, being indeterminate, we may call $x$, making it small for a "condensed" letter, and broad as need be for an "extended" form.

The word *march* would foot up $5\,U+3$, disregarding—as we would invariably—the amount the foot of the R projects beyond the main right-hand outline of the letter. In terms of $x$ this makes $5\,x+13$, as $U=x+2$. Allowing spaces of $1\frac{1}{4}$ unit width between letters adds 5 to the above, making $5\,x+18$ for the total length in terms of the I. Assuming $x$ equal to twice the unit we would have the whole word equal to twenty-eight units; and if it were to extend seven inches the width of the solid parts would therefore be one-quarter of an inch.

Where the width of a letter is not indicated it is assumed to be that of the U. The W is equal to $2\,U-1$. This relation, however, does not hold good in all alphabets.

The angular corners are drawn usually with the 45° triangle.

The guide-lines show what points of the various letters are to be found on the same level, and should be but faintly pencilled.

As remarked in Art. 27, the extended form of Gothic is one of the best for dimensioning and lettering *working drawings*, and is rapidly coming into use by the profession.

258. The Full-Block letter next illustrated is easier to work with than the Gothic in the matter of preliminary estimate, as the width of each letter—in terms of unit squares—is evident at a glance.

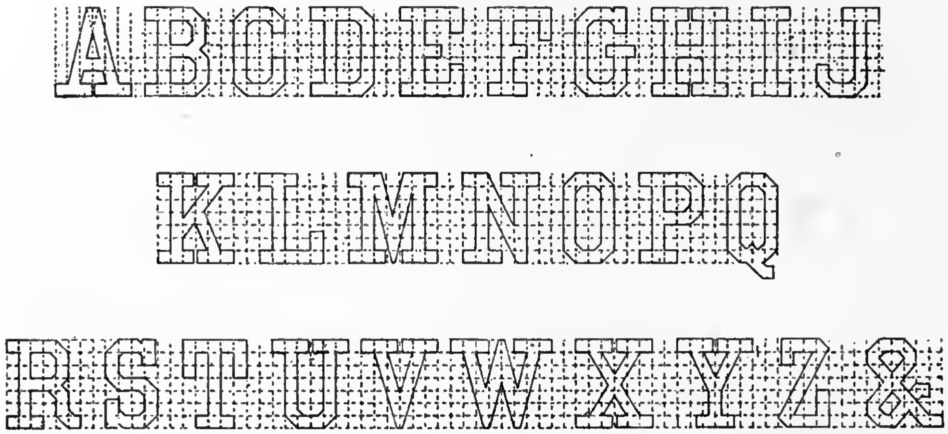
The same word *march* would foot up twenty-seven squares without allowing for spaces between letters. Calling the latter each *two* we would have thirty-five squares for the same length as before (seven inches), making one-fifth of an inch for the width of the solid parts. For convenience the widths of the various letters are summarized:

I = 3; C, G, O, Q, S, Z = 4; A, B, D, E, F, J, L, P, R, T, & = 5; H, K, N, U, V, X, Y = 6; M = 7; W = 8.

259. In case the preliminary figuring were only approximate and there were but two words in the line, as, for example, *Mechanical Drawing*, a safe method of working would be to make a fair allowance for the space between the words, begin the first word at the calculated distance to the left of the vertical centre-line, complete it, then work the second word backward, beginning with the

G as far to the right of the reference line as the M was to the left. On completing the second word any difference between the actual and the estimated length of the words, due to over- or under-width of such letters as M, W and I, will be merged into the space between the words.

Fig. 148.



With three words in a line the same method might be adopted, the middle word being easily placed half way between the others, which, by this method of construction would not only begin correctly but also terminate where they should.

260. Note particularly that the top of a B is always slightly smaller than the bottom; similarly with the S. This is made necessary by the fact that the eye seems to exaggerate the upper half of a letter. To get an idea of the amount of difference allowable compare the following equal letters printed from Roman type, condensed. Although not so important in the E, some difference between top and bottom may still to advantage be made. Another refinement is the location of the horizontal cross-bar of an A slightly below the middle of the letter.

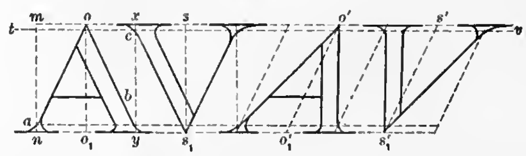
Fig. 149.

SS

261. While vertical letters are most frequently used, yet no handsomer effect can be obtained than by a well-executed inclined letter. The angle of inclination should be about 70°.

Beginners usually fail sadly in their first attempt with the A and V, one of whose sides they give the same slant as the upright of the other letters. In point of fact, however, it is the imaginary (though, in the construction, pencilled) *centre-line* which should have that inclination. See Fig. 150.

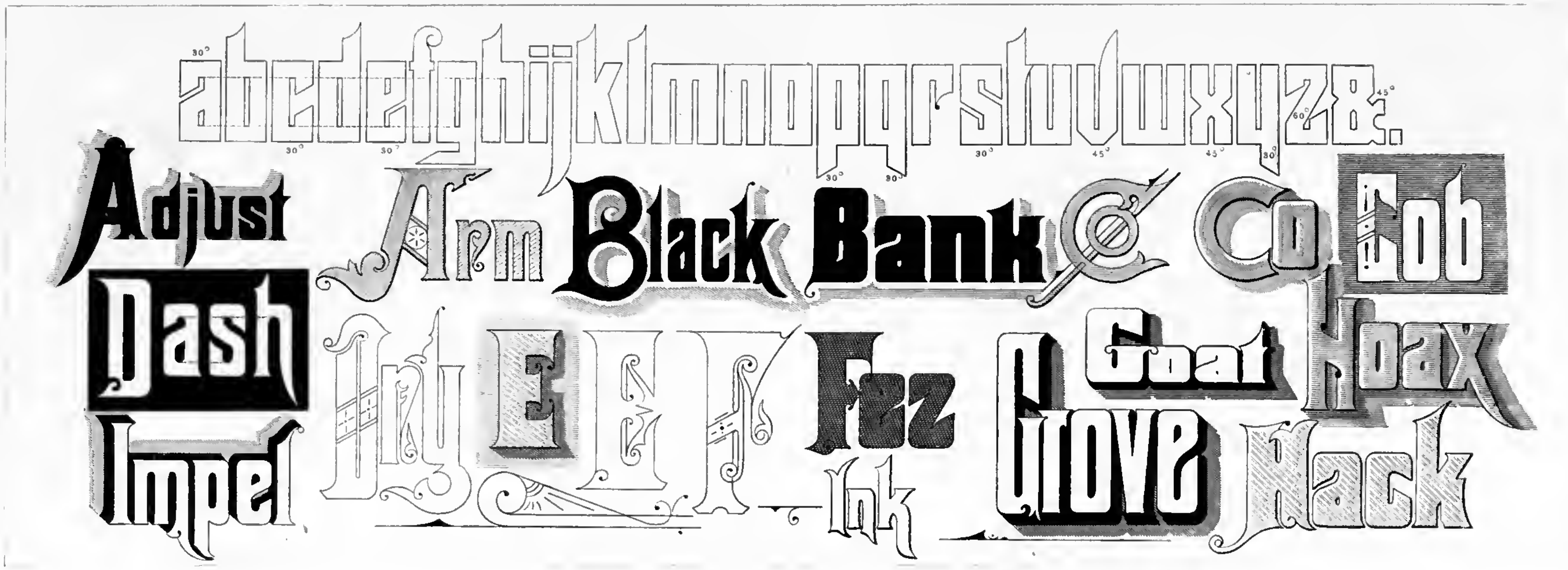
Fig. 150.



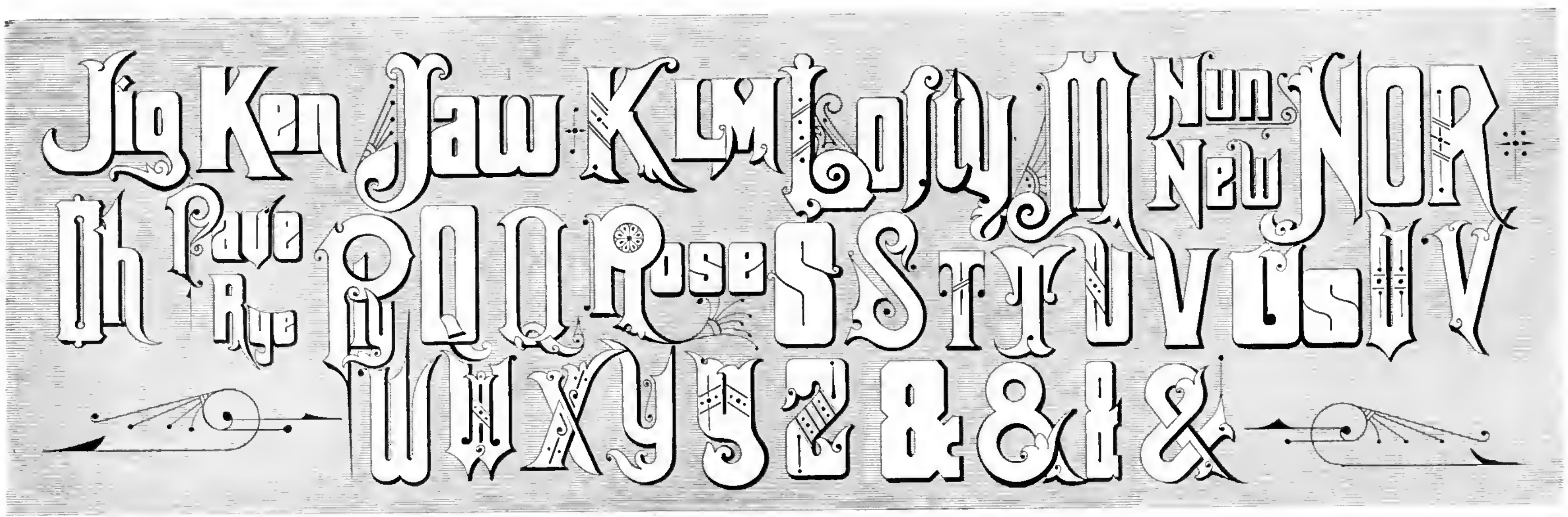
In these forms—the Roman and Italic Roman—the union of the light horizontals or "seriffs" with the other parts is in general effected by means of fine arcs, called "fillets," drawn free-hand. On many letters of this alphabet *some* lines will, however, meet at an angle, and only a careful examination of good models will enable one to construct correct forms. Upon the size of the fillets the appearance of the letter mainly depends, as will be seen by a glance at Fig. 151, which reproduces, exactly, the N of each of two leading alphabet books. If the fillets round out to the end of the spur of the letter, a coarse and bulky appearance is evidently the result; while a fine curve, leaving the straight horizontals projecting beyond them, gives the finish desired. This is further illustrated by No. 23 of the alphabets appended, a type which for clearness and elegance is a triumph of the founder's art. As usually constructed, however, the D and R are finished at the top like the P.

Fig. 151.

*NN*




RAILROAD TYPE.



262. The Roman alphabet and its inclined or *italic* form are much used in topographical work. A text-book devoted entirely to the Roman alphabet is in the market, and in some works on topographical drawing very elaborate tables of proportions for the letters are presented; these answer admirably for the construction of a standard alphabet, but in practice the proportions of the model would be preserved by the draughtsman no more closely than his eye could secure. Usually the small letters should be about three-fifths the height of the capitals. Except when more than one-third of an inch in height these letters should be entirely free-hand.

263. *When a line of a title is curved* no change is made in the *forms* of the letters; but if of a *vertical*, as distinguished from a slanting or *italic* type, the centre-line of each letter should, if produced, pass through the centre of the curve.

Italic letters, when arranged on a curve, should have their centre-lines inclined at the same angle to the normal (or radius) of the curve as they ordinarily make with the vertical.

264. An alphabet which gives a most satisfactory appearance, yet can be constructed with great rapidity, is what we may call the "Railroad" type, since the public has become familiar with it mainly from its frequent use in railroad advertisements.

The fundamental forms of the small letters, with the essential construction lines, are given in rectangular outline in the complete alphabet on the preceding page, with various modifications thereof in the words below them, showing a large number of possible effects.

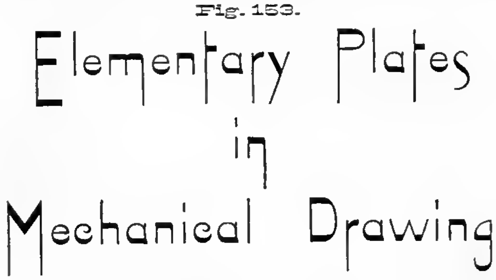
At least one plain and fancy capital of each letter is also to be found on the same page, with in some instances a still larger range of choice.

No handsomer effects are obtainable than with this alphabet, when brush tints are employed for the undertone and shadows.

Fig. 152.


265. For rapid lettering on tracing-cloth, Bristol board or any smooth-surfaced paper a style long used abroad and increasing in favor in this country is that known as *Round Writing*, illustrated by Fig. 152, and for which a special text-book and pens have been prepared by F. Soennecken. The pens are stubs of various widths, cut off obliquely, and when in use should not, as ordinarily, be dipped into the ink, but the latter should be inserted, by means of another pen, between the *top* of the Soennecken pen and the brass "feeder" that is usually slipped over it to regulate the flow.

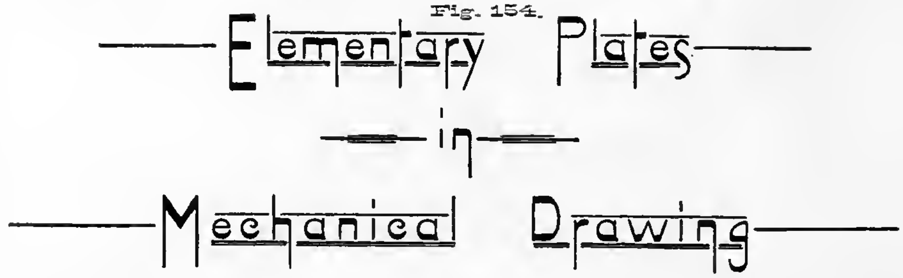
Fig. 153.

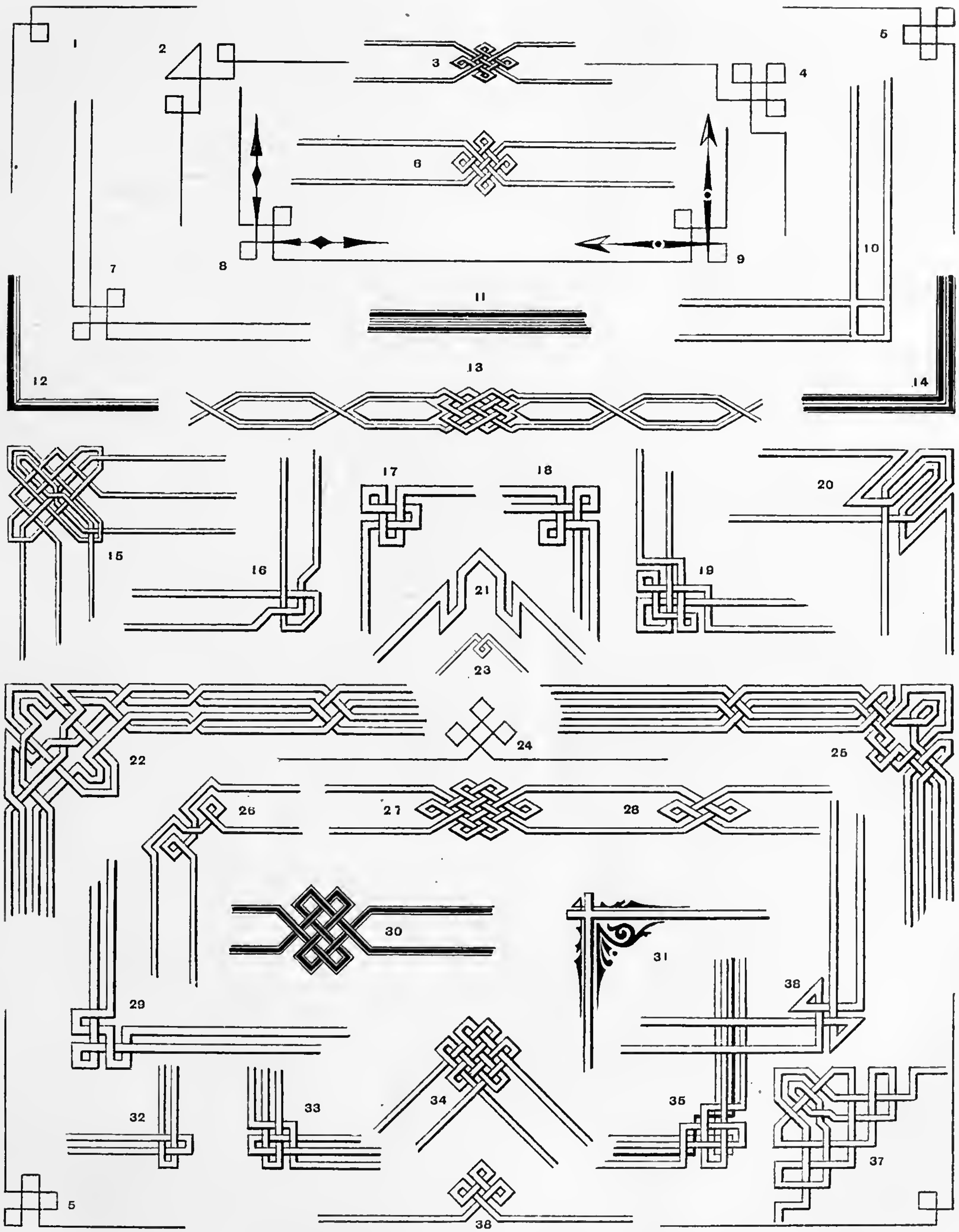



The Soennecken Round Writing Pens are also by far the best for lettering in *Old English*, *German Text* and kindred types.

The improvement due to the addition of a few straight lines to an ordinary title will become evident by comparing Figs. 153 and 154. The judicious use of "word ornaments," such as those of alphabets 33, 42, 49, and of several of the other forms illustrated, will greatly enhance the appearance of a title without materially increasing the time expended on it. This is illustrated in the lower title on page 89.

Fig. 154.

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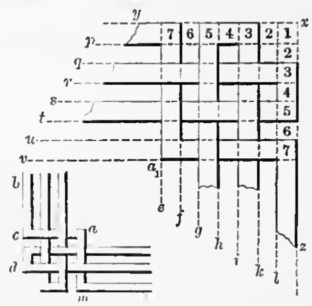
266. *Borders.* Another effective adjunct to a map or other drawing is a neat border. It should be strictly in keeping with the drawing, both as to character and simplicity.

On page 95 a large number of corner designs and borders is presented, one-third of them original designs, by the writer, for this work. The principle of their construction is illustrated by Fig. 155, in which the larger design shows the necessary preliminary lines, and the smaller the complete corner. It is evident in this, as in all cases of interlaced designs, that we must first lay off each way from the corner as many equal distances as there are bands and spaces, and lightly make a network of squares—or of rhombi, if the angles are acute—by pencilled construction-lines through the points of division.

267. *Shade lines on borders.* The usual rule as to shade lines applies equally to these designs, thus: Following any band or pair of lines making the turns as one piece, if it runs *horizontally* the *lower* line is the heavier, while in a *vertical* pair the *right-hand* line is the shaded line. This is on the assumption that the light is coming in the direction usually assumed for mechanical drawings, i. e., descending diagonally from left to right.

Fig. 155.



In case a pair of lines runs obliquely, the shaded lines may be determined by a study of their location on the designs of the plate of borders.

It need hardly be said that on any drawing and its title the light should be supposed to come from *but one direction throughout*, and not be shifted; and the shaded lines should be located accordingly. This rule is always imperative.

In drawing for scientific illustration or in art work it is allowable to depart from the usual strictly conventional direction of light, if a better effect can thereby be secured.

268. A striking letter can be made by drawing the shade line only, as in Fig. 146, page 90, which we may call "Full-Block Shade-Line," being based upon the alphabet of Fig. 148, page 92, as to construction. Owing to its having more projecting parts it gives a much handsomer effect than the

HALF-BLOCK SHADE-LINE.

The student will notice that the light comes from different directions in the two examples.

These forms are to the ordinary fully-outlined letters what art work of the "impressionist" school is to the extremely detailed and painstaking work of many; what is actually seen suggests an equal amount not on the paper or canvas.

269. While a teacher of draughting may well have on hand, as reference works for his class, such books on lettering as Prang's, Becker's and others equally elaborate, yet they will be found of only occasional service, their designs being as a rule more highly ornate than any but the specialist would dare undertake, and mainly of a character unsuitable for the usual work of the engineering or architectural draughtsman, whose needs were especially in mind when selecting types for this work.

The alphabets appended afford a large range of choice among the handsomest forms recently designed by the leading type manufacturers, also containing the best among former types; and with the "Railroad," Full-Block and Half-Block alphabets of this chapter, proportioned and drawn by the writer, supply the student with a practical "stock in trade" that it is believed will require but little, if any, supplementing.

# *CHAPTER VIII.*

## BLUE-PRINT AND OTHER COPYING PROCESSES.—METHODS OF ILLUSTRATION.

270. While in a draughting office the process described below is, at present, the only method of copying drawings with which it is *absolutely essential* that the draughtsman should be thoroughly acquainted, he may, nevertheless, find it to his advantage to know how to prepare drawings for reproduction by some of the other methods in most general use. He ought also to be able to recognize, usually, by a glance at an illustration, the method by which it was obtained. Some brief hints on these points are therefore introduced.

Obviously, however, this is not the place to give full particulars as to all these processes, even were the methods of manipulation not, in some cases, still "trade secrets"; but the important details concerning them, that have become common property, may be obtained from the following valuable works: *Modern Heliographic Processes,** by Ernst Lietze; *Photo-Engraving, Etching and Lithography,*† by W. T. Wilkinson; *Modern Reproductive Graphic Processes,** by Jas. S. Pettit, and *Photo-Engraving*, by Carl Schraubstadter, Jr.

### THE BLUE-PRINT PROCESS.

271. By means of this process, invented by Sir John Herschel, any number of copies of a drawing can be made, in white lines on a blue ground. In Arts. 43 and 45 some hints will be found as to the relative merits of tracing-cloth and "Bond" paper, for the original drawing.

A sheet of paper may be sensitized to the action of light by coating its surface with a solution of red prussiate of potash (ferrocyanide of potassium) and a ferric salt. The chemical action of light upon this is the production of a ferrous salt from the ferric compound; this combines with the ferrocyanide to produce the final blue undertone of the sheet; while the portions of the paper from which the light was intercepted by the inked lines, become white after immersion in water.

The proportions in which the chemicals are to be mixed are, apparently, a matter of indifference, so great is the disparity between the recipes of different writers; indeed, one successful draughtsman says: "Almost any proportion of chemicals will make blue-prints." Whichever recipe is adopted—and a considerable range of choice will be found in this chapter—the hints immediately following are of general application.

272. Any white paper will do for sensitizing that has a hard finish, like that of ledger paper, so as not to absorb the chemical solution.

To sensitize the paper dissolve the ferric salt and the ferrocyanide in water, separately, as they are then not sensitive to the action of light. The solutions should be *mixed* and *applied* to the paper only in a *dark room.*

Although there is the highest authority for "floating the paper to be sensitized for two minutes on the surface of the liquid," yet the best American practice is to apply the solution with a soft flat brush about four inches wide. The main object is to obtain an even coat, which may usually

*Published by the D. Van Nostrand Company, New York. †American Edition revised and published by Edward L. Wilson, New York.

be secured by a primary coat of horizontal strokes followed by an overlay of vertical strokes; the second coat applied before the first dries. If necessary, another coat of diagonal strokes may be given to secure evenness. The thicker the coating given the longer the time required in printing. A bowl or flat dish or plate will be found convenient for holding the small portion of the solution required for use at any one time. The chemicals should not get on the back of the sheet.

Each sheet, as coated, should be set in a dark place to dry, either "tacked to a board by two adjacent corners," or "hung on a rack or over a rod," or "placed in a drawer—one sheet in a drawer,"—varying instructions, illustrating the quite general truth that there are usually several almost equally good ways of doing a thing.

273. To copy a drawing, place the prepared paper, sensitized side up, on a drawing-board or printing-frame on which there has been fastened, *smoothly*, either a felt pad or canton flannel cloth. The drawing is then immediately placed over the first sheet, inked side up, and contact secured between the two by a large sheet of plate glass, placed over all.

Exposure in the direct rays of the sun for four or five minutes is usually sufficient. The progress of the chemical action can be observed by allowing a corner of the paper to project beyond the glass. It has a grayish hue when sufficiently exposed.

If the sun's rays are not direct, or if the day is cloudy, a proportionately longer time is required, running up in the latter case, from minutes into hours. Only experiment will show whether one's solution is "quick" or "slow;" or the time required by the degree of cloudiness.

A solution will print more quickly if the amount of water in it be increased, or if more iron is used; but in the former case the print will not be as dark, while in the latter the results, as to whiteness of lines, are not so apt to be satisfactory.

Although fair results can be obtained with paper a month or more after it has been sensitized, yet they are far more satisfactory if the paper is prepared each time (and dried) just before using.

On taking the print out of the frame it should be immediately immersed and thoroughly washed in *cold water* for from three to ten minutes, after which it may be dried in either of the ways previously suggested.

If many prints are being made, the water should be frequently changed so as not to become charged with the solution.

274. The entire process, while exceedingly simple in theory, varies, as to its results, with the experience and judgment of the manipulator. To his choice the decision is left between the following standard recipes for preparing the sensitizing solution. The "parts" given are all by weight. In every case the potash should be pulverized, to facilitate its dissolving.

No. 1. (From *Le Génie Civil.*)

| | | |
|---|---|---|
| *Solution No. 1.* | Red Prussiate of Potash | 8 parts. |
| | Water | 70 parts. |
| *Solution No. 2.* | Citrate of Iron and Ammonia | 10 parts. |
| | Water | 70 parts. |

Filter the solutions separately, mix equal quantities and then filter again.

No. 2. (From U. S. Laboratory at Willett's Point).

| | | |
|---|---|---|
| *Solution No. 1.* | Double Citrate of Iron and Ammonia | 1 ounce. |
| | Water | 4 ounces. |
| *Solution No. 2.* | Red Prussiate of Potassium | 1 ounce. |
| | Water | 4 ounces. |

No. 3. (Lietze's Method).

*Stock Solution.* { 5 ounces, avoirdupois, Red Prussiate of Potash.
32 fluid ounces....................................Water.

"After the red prussiate of potash has been dissolved—which requires from one to two days—the liquid is filtered. This solution remains in good condition for a long time. Whenever it is required to sensitize paper, dissolve, for every two hundred and forty square feet of paper,

{ 1 ounce, avoirdupois, Citrate of Iron and Ammonia,
4½ fluid ounces.........................................Water,

and mix this with an equal volume of the stock solution.

The reason for making a stock solution of the red prussiate of potash is, that it takes a considerable time to dissolve and because it must be filtered. There are many impurities in this chemical which can be removed by filtering. Without filtering, the solution will not look clear. The reason for making no stock solution of the ferric citrate of ammonia is that such solution soon becomes moldy and unfit for use. This ferric salt is brought into the market in a very pure state, and does not need to be filtered after being dissolved. It dissolves very rapidly. In the solid form it may be preserved for an unlimited time, if kept in a well-stoppered bottle and protected against the moisture of the atmosphere. A solution of this salt, or a mixture of it with the solution of red prussiate of potash, will remain in a serviceable condition for a number of days, but it will spoil, sooner or later, according to atmospheric conditions. . . . Four ounces of sensitizing solution, for blue prints, are amply sufficient for coating one hundred square feet of paper, and cost about six cents."

For copying tracings in *blue* lines or *black* on a *white ground*, one may either employ the recipes given in Lietze's and Pettit's work, or obtain paper already sensitized, from the leading dealers in draughtsmen's supplies. *The latter course has become quite as economical, also, for the ordinary blue-print,* as the preparing of one's own supply.

For copying a drawing in *any* desired color the following method, known as *Tilhet's*, is said to give good results: "The paper on which the copy is to appear is first dipped in a bath consisting of 30 parts of white soap, 30 parts of alum, 40 parts of English glue, 10 parts of albumen, 2 parts of glacial acetic acid, 10 parts of alcohol of 60°, and 500 parts of water. It is afterward put into a second bath, which contains 50 parts of burnt umber ground in alcohol, 20 parts of lampblack, 10 parts of English glue, and 10 parts of bichromate of potash in 500 parts of water. They are now sensitive to light, and must, therefore, be preserved in the dark. In preparing paper to make the positive print another bath is made just like the first one, except that lampblack is substituted for the burnt umber. To obtain colored positives the black is replaced by some red, blue or other pigment.

In making the copy the drawing to be copied is put in a photographic printing frame, and the negative paper laid on it, and then exposed in the usual manner. In clear weather an illumination of two minutes will suffice. After the exposure the negative is put in water to develop it, and the drawing will appear in white on a dark ground; in other words, it is a negative or reversed picture. The paper is then dried and a positive made from it by placing it on the glass of a printing-frame, and laying the positive paper upon it and exposing as before. After placing the frame in the sun for two minutes the positive is taken out and put in water. The black dissolves off without the necessity of moving back and forth."

### PHOTO-AND OTHER PROCESSES.

275. If a drawing is to be reproduced on a different scale from that of the original, some one of the processes which admits of the use of the camera is usually employed. Those of most importance to the draughtsman are (1) *wood engraving;* (2) the "wax process" or *cerography;* (3) *lithography*, and (4) the various methods in which the photographic negative is made on a film of gelatine which is then used *directly*—to print from, or *indirectly*—in obtaining a metal plate from which the impressions are taken.

In the first three named above the use of the camera is not invariably an element of the process.

All under the fourth head are essentially photo-processes and their already large number is constantly increasing. Among them may be mentioned *photogravure*, *collotype*, *phototype*, *autotype*, *photoglyph*, *albertype*, *heliotype*, and *heliogravure.*

### WOOD ENGRAVING.

276. There is probably no process that surpasses the best work of skilled engravers on wood. This statement will be sustained by a glance at Figs. 14, 15, 20–24, 134, 136, and those illustrating mathematical surfaces, in the next chapter. Its expensiveness, and the time required to make an illustration by this method, are its only disadvantages.

Although the camera is often employed to transfer the drawing to the boxwood block in which the lines are to be cut, yet the original drawing is quite as frequently made *in reverse*, directly on the block, by a professional draughtsman who is supposed to have at his disposal either the object to be drawn or a photograph or drawing thereof. The outlines are pencilled on the block, and the shades and shadows given in brush tints of India ink, re-enforced, in some cases, by the pencil, for the deepest shadows.

The "high lights" are brought out by Chinese white. A medium wash of the latter is also usually spread upon the block as a general preliminary to outlining and shading.

The task of the engraver is to reproduce faithfully the most delicate as well as the strongest effects obtained on the block with pencil and brush, cutting away all that is not to appear in black in the print. The finished block may then be used to print from directly, or an electrotype block can be obtained from it which will stand a large number of impressions much better than the wood.

### CEROGRAPHY.

277. For map-making, illustrations of machinery, geometrical diagrams and all work mainly in straight lines or simple curves, and not involving too delicate gradations, the cerographic or "wax process" is much employed. For clearness it is scarcely surpassed by steel engraving. Figures 36, 90 and 107 are good specimens of the effects obtainable by this method. The successive steps in the process are (a) the laying of a thin, even coat of wax over a copper plate; (b) the transfer of the drawing to the surface of the wax, either by tracing or—more generally—by photography; (c) the re-drawing or rather the cutting of these lines in the wax, the stylus removing the latter to the surface of the copper; (d) the taking of an electrotype from the plate and wax, the deposit of copper filling in the lines from which the wax was removed.

Although in the preparation of the original drawing the lines may preferably be inked, yet it is not absolutely necessary, provided a pencil of medium grade be employed.

Any letters desired on the final plate may be also pencilled in their proper places, as the engraver makes them on the wax with type.

A surface on which section-lining or cross-hatching is desired may have that fact indicated upon it in writing, the direction and number of lines to the inch being given. Such work is then done with a ruling machine.

Errors may readily be corrected, as the surface of the wax may be made smooth, for recutting, by passing a hot iron over it.

### LITHOGRAPHY.—PHOTO-LITHOGRAPHY.—CHROMO-LITHOGRAPHY.

278. For *lithographic processes* a fine-grained, imported limestone is used. The drawing is made with a greasy ink—known as "lithographic"—upon a specially prepared paper, from which it is transferred under pressure, to the surface of the stone. The un-inked parts of the stone are kept thoroughly moistened with water, which prevents the printer's ink (owing to the grease which the latter contains) from adhering to any portion except that from which the impressions are desired.

*Photo-lithography* is simply lithography, with the camera as an adjunct. The positive might be made directly upon the surface of the stone by coating the latter with a sensitizing solution; but, in general, for convenience, a sensitized gelatine film is exposed under the negative, and by subsequent treatment gives an image in relief which, after inking, can be transferred to the surface of the stone as in the ordinary process.

*Chromo-lithography*, or lithography in colors, has been a very expensive process, owing to its requiring a separate stone for each color. Recent inventions render it probable that it will be much simplified, and the expense correspondingly reduced. The details of manipulation are closely analogous to those of ink prints.

When colored plates are wanted, in which delicate gradations shall be indicated, chromo-lithography may preferably be adopted; although "half-tones," with colored inks, give a scarcely less pleasing effect, as illustrated by Figs. 7–10, Plate II. But for simple line-work, in two or more colors, one may preferably employ either cerography or photo-engraving, each of which has not only an advantage, as to expense, over any lithographic process, but also this in addition—that the blocks can be used by any printer; whereas lithographing establishments necessarily not only prepare the stone but also do the printing.

### PHOTO-ENGRAVING.—PHOTO-ZINCOGRAPHY.

279. In this popular and rapid process a sensitized solution is spread upon a smooth sheet of zinc, and over this the photographic negative is placed. Where not acted on by the light the coating remains soluble and is washed away, exposing the metal, which is then further acted on by acids to give more *relief* to the remaining portions.

Except as described in Art. 281 this process is only adapted to inked work in lines or dots, which is reproduced faithfully, to the smallest detail. Among the best photo-engravings in this book are Figs. 10 and 11, 50, 79 and 80.

280. The following instructions for the preparation of drawings, for reproduction by this process, are those of the American Society of Mechanical Engineers as to the illustration of papers by its members, and are, in general, such as all the engraving companies furnish on application.

"All lines, letters and figures must be *perfectly black* on a white ground. Blue prints are not available, and red figures and lines will not appear. The smoother the paper, and the blacker the ink, the better are the results. Tracing-cloth or paper answers very well, but rough paper—even

Whatman's—gives bad lines. India ink, ground or in solution, should be used; and the best lines are made on Bristol board, or its equivalent with an enameled surface. Brush work, in tint or grading, unfits a drawing for immediate use, since only line work can be photographed. Hatching for sections need not be completed in the originals, as it can be done easily by machine on the block. If draughtsmen will indicate their sections unmistakably, they will be properly lined, and tints and shadows will be similarly treated.

The best results may be expected by using an original twice the height and width of the proposed block. The reduction can be greater, provided care has been taken to have the lines far enough apart, so as not to mass them together. Lines in the plate may run from 70 to 100 to the inch, and there should be but half as many in a drawing which is to be reduced one-half; other reductions will be in like proportion.

Draughtsmen may use photographic prints from the objects if they will go over with a carbon ink all the lines which they wish reproduced. The photographic color can be bleached away by flowing a solution of bi-chloride of mercury in alcohol over the print, leaving the pen lines only. Use half an ounce of the salt to a pint of alcohol.

Finally, lettering and figures are most satisfactorily printed from type. Draughtsmen's best efforts are usually thus excelled. Such letters and figures had therefore best be left in pencil on the drawings, so they will not photograph but may serve to show what type should be inserted."

To the above hints should be added a caution as to the use of the rubber. It is likely to diminish the intensity of lines already made and to affect their sharpness; also to make it more difficult to draw clear-cut lines wherever it has been used.

It may be remarked with regard to the foregoing instructions that they aim at securing that uniformity, as to general appearance, which is usually quite an object in illustration. But where the preservation of the individuality and general characteristics of one's work is of any importance whatever, the draughtsman is advised to letter his own drawings and in fact finish them entirely, himself, with, perhaps, the single exception of section-lining, which may be quickly done by means of *Day's Rapid Shading Mediums* or by other technical processes.

281. *Half Tones.* Photo-zincography may be employed for reproducing delicate gradations of light and shade, by breaking up the latter when making the photographic negative. The result is called a *half tone*, and it is one of the favorite processes for high-grade illustration. Figs. 95 and 130 illustrate the effects it gives. On close inspection a series of fine dots in regular order will be noticed, or else a net-work, so that no tone exists unbroken, but all have more or less white in them.

The methods of breaking up a tone are very numerous. The first patent dates back to 1852. The principle is practically the same in all, viz., between the object to be photographed and the plate on which the negative is to be made there is interposed a "screen" or sheet of thin glass, on which the desired mesh has been previously photographed.

In the making of the "screen" lies the main difference between the variously-named methods. In Meissenbach's method, by which Figs. 95 and 130 were made, a photograph is first taken, on the "screen," of a pane of clear glass in which a system of parallel lines—one hundred and fifty to the inch—has been cut with a diamond. The ruled glass is then turned at right angles to its first position and its lines photographed on the screen over the first set, the times of exposure differing slightly in the two cases, being generally about as 2 to 3.

This process is well adapted to the reproduction of "wash" or brush-tinted drawings, photographs, etc. The object to be represented, if small, may preferably be furnished to the engraving company and they will photograph it direct.

GELATINE FILM PHOTO-PROCESSES.

282. As stated in Art. 275, in which a few of the above processes are named, a gelatine film may be employed, either as an adjunct in a method resulting in a metal block, or to print from directly; in the latter case the prints must be made, on special paper, by the company preparing the film. In the composition and manipulation of the film lies the main difference between otherwise closely analogous processes. For any of them the company should be supplied with either the original object or a good drawing or photographic negative thereof.

Not to unduly prolong this chapter—which any sharp distinction between the various methods would involve, yet to give an idea of the general principles of a gelatine process we may conclude with the details of the preparation of a *heliotype* plate, given in the language of the circular of a leading illustrating company. Figs. 1—5 of Plate II illustrate the effect obtained by it.

"Ordinary cooking gelatine forms the basis of the positive plate, the other ingredients being bichromate of potash and chrome alum. It is a peculiarity of gelatine, in its normal condition, that it will absorb *cold water*, and swell or expand under its influence, but that it will dissolve in *hot water*. In the preparation of the plate, therefore, the three ingredients just named, being combined in suitable proportions, are dissolved in hot water, and the solution is poured upon a level plate of glass or metal, and left there to dry. When dry it is about as thick as an ordinary sheet of parchment, and is stripped from the drying-plate, and placed in contact with the previously-prepared negative, and the two together are exposed to the light. The presence of the bichromate of potash renders the gelatine sheet sensitive to the action of light; and wherever light reaches it, the plate, which was at first gelatinous or absorbent of water, becomes leathery or waterproof. In other words, wherever light reaches the plate, it produces in it a change similar to that which tanning produces upon hides in converting them into leather. Now it must be understood that the negative is made up of transparent parts and opaque parts; the transparent parts admitting the passage of light through them, and the opaque parts excluding it. When the gelatine plate and the negative are placed in contact, they are exposed to light with the negative *uppermost*, so that the light acts through the translucent portions, and waterproofs the gelatine underneath them; while the opaque portions of the negative shield the gelatine underneath them from the light, and consequently those parts of the plate remain unaltered in character. The result is a thin, flexible sheet of gelatine of which a portion is waterproofed, and the other portion is absorbent of water, the waterproofed portion being the image which we wish to reproduce. Now we all know the repulsion which exists between water and any form of *grease*. Printer's ink is merely grease united with coloring-matter. It follows, that our gelatine sheet, having water applied to it, will absorb the water in its unchanged parts; and, if ink is then rolled over it, the ink will adhere only to the waterproofed or altered parts. This flexible sheet of gelatine, then, prepared as we have seen, and having had the image impressed upon it, becomes the *heliotype plate*, capable of being attached to the bed of an ordinary printing-press, and printed in the ordinary manner. Of course, such a sheet must have a solid base given to it, which will hold it firmly on the bed of the press while printing. This is accomplished by uniting it, under water, with a metallic plate, exhausting the air between the two surfaces, and attaching them by atmospheric pressure. The plate, with the printing surface of gelatine attached, is then placed on an ordinary platen printing-press, and inked up with ordinary ink. A mask of paper is used to secure white margins for the prints; and the impression is then made, and is ready for issue."

"*The study of Descriptive Geometry possesses an important philosophical peculiarity, quite independent of its high industrial utility. This is the advantage which it so pre-eminently offers in habituating the mind to consider very complicated geometrical combinations in space, and to follow with precision their continual correspondence with the figures which are actually traced—of thus exercising to the utmost, in the most certain and precise manner, that important faculty of the human mind which is properly called 'imagination,' and which consists, in its elementary and positive acceptation, in representing to ourselves, clearly and easily, a vast and variable collection of ideal objects, as if they were really before us. . . . . While it belongs to the geometry of the ancients by the character of its solutions, on the other hand it approaches the geometry of the moderns by the nature of the questions which compose it. These questions are in fact eminently remarkable for that generality which constitutes the true fundamental character of modern geometry; for the methods used are always conceived as applicable to any figures whatever, the peculiarity of each having only a purely secondary influence.*"

AUGUSTE COMTE: Cours de Philosophie Positive.

"*A mathematical problem may usually be attacked by what is termed in military parlance the method of 'systematic approach;' that is to say, its solution may be gradually felt for, even though the successive steps leading to that solution cannot be clearly foreseen. But a Descriptive Geometry problem must be seen through and through before it can be attempted. The entire scope of its conditions as well as each step toward its solution must be grasped by the imagination. It must be 'taken by assault.'*"

GEORGE SYDENHAM CLARKE, Captain, Royal Engineers.

# CHAPTER IX.

## ORTHOGRAPHIC PROJECTION UPON MUTUALLY PERPENDICULAR PLANES.—DEFINITION, CLASSIFICATION AND ILLUSTRATION OF MATHEMATICAL LINES AND SURFACES.

283. In this and nearly all the later chapters of this work the principles of what has been generally known as *Descriptive Geometry* are either examined or applied.

In Art. 19—which, with Arts. 2, 3 and 14, should be reviewed at this point—reasons are given for calling this science *Monge's Descriptive.* Certain German writers call it *Monge's Orthogonal Projection.* The popular titles *Mechanical Drawing, Practical Solid Geometry, Orthographic Projection,* etc., are usually merely indicative of more or less restricted applications of Monge's Descriptive to some special industrial arts; and *working drawings* of bridge and roof trusses, machinery, masonry and other constructions, are simply accurately scaled and fully dimensioned projections made in accordance with its principles.

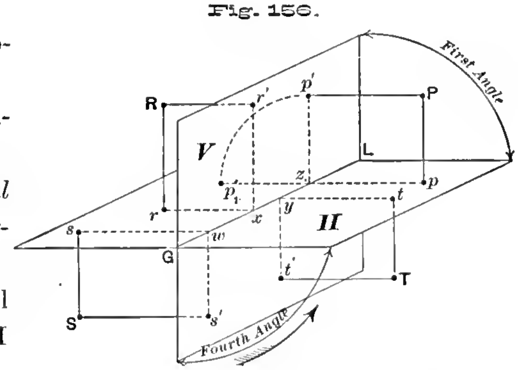
Monge's service to mathematics and graphical science, which, according to Chasles*, inaugurated the fifth epoch in geometrical history, consisted, not in inventing the method of representing objects by their projections—for with that the ancients were thoroughly familiar, but in perceiving and giving scientific form to the principles and theorems which were fundamental to the special solutions of a great number of graphical problems handed down through many centuries, and many of which had been the monopoly of the Freemasons. Emphasizing in this chapter the abstract principles of the subject, treating it as a pure science, and giving a general outline of the field of its application, its more practical and commercial side is left for the next chapter, including the modifications in vogue in the draughting offices of leading mechanical engineers.

Statements without proof are given whenever their truth is reasonably self-evident.

### FUNDAMENTAL PRINCIPLES.

284. *The orthographic projection of a point on a plane* is the foot of the perpendicular from the point to the plane.

Fig. 156.




In Fig. 156 the perpendiculars $Pp'$ and $Pp$ give the projections, $p'$ and $p$, of the point $P$.

The planes of projection are shown in their space position; one, H, horizontal, the other, V, vertical.

The projection, $p$, on H, is called the *plan* or *horizontal projection,* (h. p.) of $P$. The point $p'$ is the *elevation* or *vertical projection* (v. p.) of P.

Projections on the vertical plane are denoted by small letters with a single accent or "prime." Projections on H are small letters unaccented.

A point may be named by its space-letter, the capital, or by its projections; thus, we may speak of the point $P$ or of the point $pp'$.

*Aperçu Historique sur l'Origin et le Developpement des Methodes en Géométrie.

We shall call $Pp$ the H-*projector* of $P$, since it gives the projection of $P$ on H. Similarly, $Pp'$ is the V-*projector* of $P$. A *projector-plane* is then the plane containing both projectors of a point, and is evidently perpendicular to both V and H by virtue of containing a line perpendicular to each.

285. V and H intersect in a line called the *ground line*, hereafter denoted by G. L. They make with each other four diedral angles.

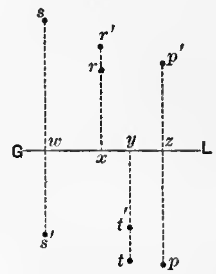
The observer is always supposed to be in the *first angle*, viz., that which is above H and in front of V. We shall call it $Q_1$, or the *first quadrant*. $Q_2$ is then the second quadrant, back of V but above H. $Q_3$ is below $Q_2$, while $Q_4$ is immediately below the first angle.

Points $R$ and $S$, in the second and third quadrants respectively, have their elevations, $r'$ and $s'$, on opposite sides of G. L., while their plans, $r$ and $s$, are on the back half of H.

The point $T$, in $Q_4$, has its v. p. at $t'$, below G. L., and its h. p. at $t$, in front of G. L.

286. *Transition from pictorial to orthographic view.—Rotation.* In making a drawing in the ordinary way, and not pictorially, we suppose the planes V and H brought into coincidence by revolution about G. L., the upper part of V uniting with the back part of H, while lower V and front H merge in one. The arrow (Fig. 156) shows such direction of revolution, after which any vertical projection, as $p'$, is found at $p'_1$, on a line $pz$ *perpendicular to* G. L. and containing the plan $p$. This is inevitable, from the following consideration: Any point when revolved about an axis describes a circle whose centre is on the axis and whose plane is perpendicular to the axis; but as the projector-plane of $P$ contains $p'$—the point to be revolved, and is perpendicular to G. L. (the axis) because perpendicular to both H and V, it must be the plane of rotation of $p'$, which can therefore only come into H somewhere on $pz$ (produced).

Fig. 157.



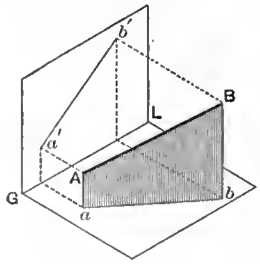

In Fig. 157 we find Fig. 156 represented in the ordinary way. Only the projections of the points appear. V and H are, as usual, considered indefinite in extent, and their boundaries have disappeared. A projector-plane is shown only by the line, perpendicular to G. L., in which its intersections with V and H coincide.

287. A point ($pp'$) in the *first* angle has its h. p. below G. L., and its v. p. above. For the *third* angle the reverse is the case, the *plan*, $s$, *above*, and the *elevation*, $s'$, *below*. The second and fourth angles are also opposites, both projections, $rr'$, being above G. L. for the former, and both, $tt'$, below for the latter.

288. For *any* angle *the actual distance of a point from* H, as shown by any H-projector $Pp$ (Fig. 156), is equal to $p'z$ (either figure)—the distance of the v. p. of the point from G. L. Similarly, the V-projector of a point, as $Rr'$ (Fig. 156), showing the actual distance of a point from V, is equal to the distance of the h. p. of the point from G. L.

*If one projection of a point is on* G. L. the point is *in* a plane of projection. If in H, the *elevation* of the point will be on G. L.; similarly, the *plan* is on G. L. if the point lies in V.

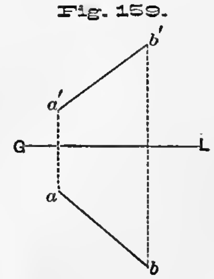
Fig. 158.




289. *The projection of a line* is the line containing the projections of all its points. The projection of a straight line will be a straight line; for its extremity-projectors, as $Aa$ and $Bb$ (Fig. 158) would determine a plane perpendicular to H and containing $AB$; in such plane all other H-projectors must lie; hence meet H in $ab$, which is straight because the intersection of two planes.

In Fig. 159 we see the line $AB$ of Fig. 158, orthographically represented.

Fig. 159.



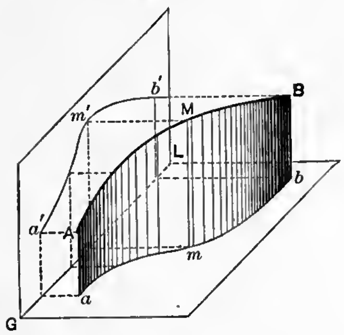

290. *Projecting planes.—Traces.* A plane containing the projectors of a straight line is called a *projecting plane* of the line. ($ABba$, Fig. 158).

A V-*projecting plane* of a line contains the line and is perpendicular to the vertical plane.

The H-*projecting plane* of a line is the vertical plane containing it.

*Traces.* The intersection of a surface by a given line or surface is called a *trace.* The trace of a *line* is a *point;* of a *surface* is a *line.* If on H it is called a *horizontal trace* (*h.t.*); on V, a *vertical trace,* abbreviated to *v.t.*

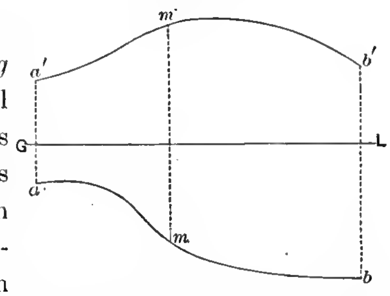
Fig. 160.




291. *The projection of a curve* is in general a curve, the *trace*—upon V or H—of the cylindrical surface* whose elements are the projectors of the points of the curve. (See Figs. 160 and 161).

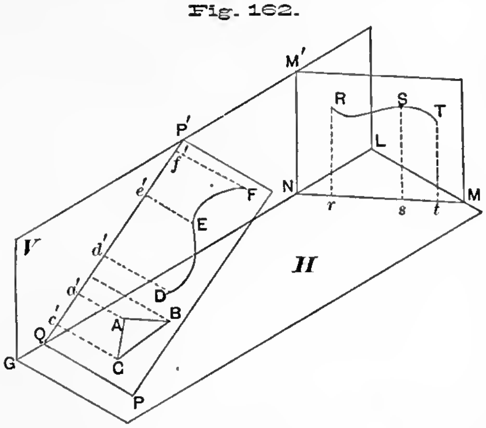
*The projection of a plane curve* will be equal and parallel to the original curve, when the latter is parallel to the plane on which it is projected.

Fig. 161.




292. *All lines,* straight or curved, *lying in a plane that is perpendicular to* V, will be projected on V in the v.t. of the plane. A similar remark applies to the plans of lines lying in a *vertical* plane. Fig. 162 illustrates these statements. The plane $P'QP$, being perpendicular to V—as shown by the h.t. ($PQ$) being perpendicular to G.L—all points in the plane, as $A, B, C, D, E, F,$ will have their projections on V in the trace $P'Q$.

Fig. 162.

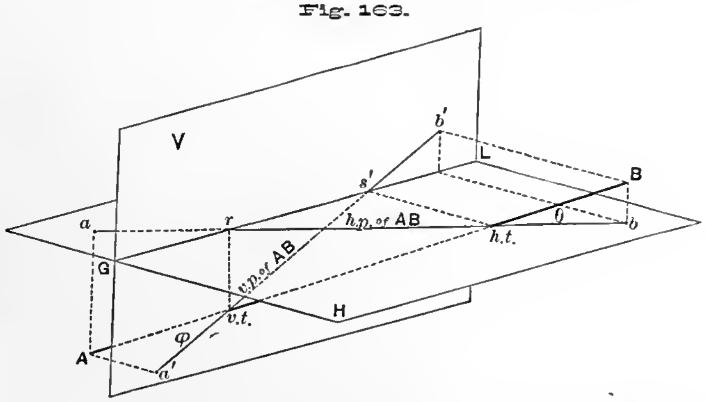



The plane $M'NM$ is vertical, since $MN$—its *vertical* trace—is perpendicular to G.L.; its h.t. ($NM$) therefore contains the h.p. of every point in the plane.

*True size of figures in planes not parallel to* V *or* H. Since the triangle $ABC$ and the curve $DEF$ are not parallel to H or V their exact *size* and *shape* would not be shown by their projections; these could, however, be readily obtained by rotating their plane into H, about the trace $PQ$, or into V about $P'Q$. Such rotation, called *rabatment*†, is described in detail in Art. 306.

293. *Traces of lines.* Fig. 163 shows that the h.t. of a line $AB$ is at the intersection of the plan $ab$ by a perpendicular to G.L. from $s'$, where the elevation $a'b'$ crosses G.L.

Fig. 163.




Similarly, the v.t. of the line is on its v.p., immediately below $r$, the intersection of the ground line by the plan $ab$; hence the rule: *To find the horizontal trace of a line* prolong the vertical projection until it meets the ground line; thence draw a perpendicular to G.L. to meet the plan of the line. An analogous construction gives the vertical trace of the line.

Fig. 164.

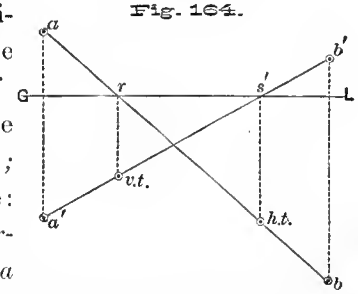



Fig. 164 illustrates by the ordinary method the same projections and traces as in Fig. 163.

*See Remark, Art. 8.

†From the French *rabattement.*

294. *The angle between a line and* H *or* V. The inclination, $\theta$, of a line to H, is that of the line to its plan $a\,b$. The angle ($\phi$) between a line and V, is that between it and its v.p., $a'\,b'$.

Fig. 165.

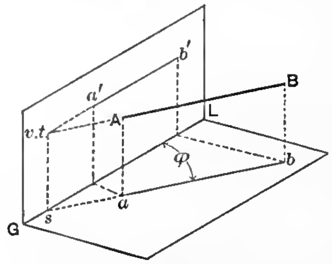
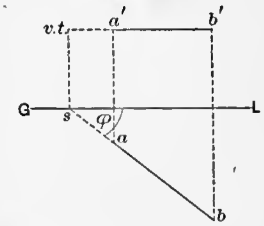



Fig. 166.




295. Any *horizontal* line has a plan, $a\,b$ (Fig. 165), equal and parallel to itself. Its elevation, $a'\,b'$, is *parallel* to G. L., and at the same distance from it as the line from H. The line makes the same angle with V that its plan makes with G. L. Such line can evidently have no h. t. Its v. t. would be found by the rule in the preceding article. In the extreme case of perpendicularity to V the v. p. of $A\,B$ would reduce to a point.

296. *A line parallel to* V *but oblique to* H has its h. p. parallel to the ground line; makes with H the same angle that its v. p. does with G. L.; equals its v. p., and has its h. t. found by the usual rule. (Fig. 167).

Fig. 167.

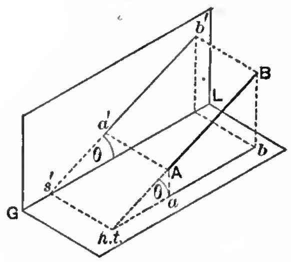
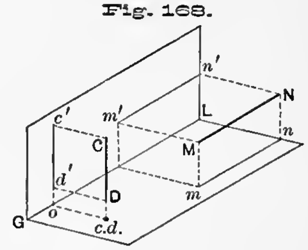



Fig. 168.

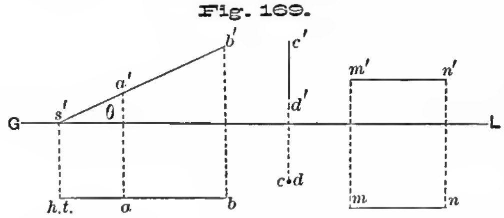



297. *A vertical line* has no v. t.; is projected on H in a point; has its v. p. parallel and equal to itself. (See $C\,D$, Fig. 168).

*A line parallel to both* V *and* H is parallel to their intersection, has no traces, and each projection equals the line. (See $M\,N$, Fig. 168).

Fig. 169 shows, orthographically, the lines $A\,B$, $C\,D$ and $M\,N$ of the two preceding figures.

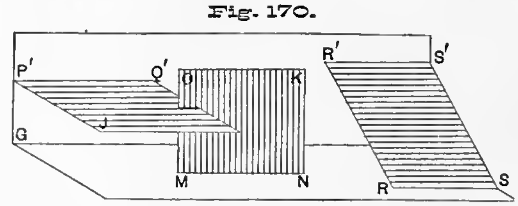
Fig. 169.




298. *Representation of planes.* A plane is represented by its traces. Like H and V, any plane is considered indefinite in extent when drawn in the usual way; though our pictorial diagrams show them bounded, to add to the appearance.

A *horizontal* plane has but one trace, that on V. A plane parallel to V, as $M\,N\,K\,O$, Fig. 170, has no vertical trace, and its horizontal trace $M\,N$ is parallel to G. L.

Fig. 170.




*A plane will have parallel traces* when it is parallel to G. L. but oblique to both H and V. ($R\,S$, $R'\,S'$, Fig. 170). Fig. 171 shows the planes of Fig. 170, as usually represented.

The traces of a plane not parallel to G. L. must meet at the same point on G. L.

299. *Planes perpendicular to a plane of projection.* If a plane is perpendicular to a plane of projection its trace on the other plane is perpendicular to G. L. This is the only case in which the angles between the ground line and the traces of a plane equal the dihedral angles made with H and V by the plane. Such equality may be thus shown: The dihedral angle between two planes equals the plane angle between two lines, cut from the planes by a third plane perpendicular to both; plane $P'\,Q\,R$ (Fig. 172) is by hypothesis perpendicular to V; hence the ground line and $P'\,Q$ are the lines cut by plane V from two other planes to which it is perpendicular; and their angle $\theta$ is, therefore, the measure of the dihedral angle between H and $P'\,Q\,R$. Similar reasoning applies to $N'\,S\,N$.

Fig. 171.

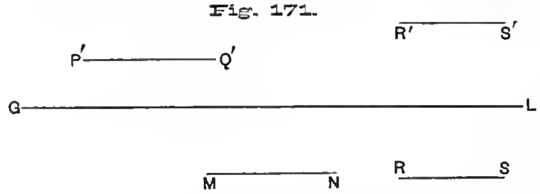
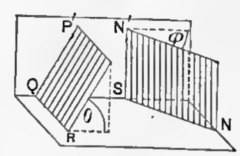



Fig. 172.



300. *Plane determined by lines.—Lines drawn in a plane.* Fig. 173 illustrates all the possibilities.

(a) *Any line parallel to* H is necessarily a horizontal line; but when also contained by a plane it is called *a horizontal* of the plane. It is obviously parallel to the h.t. ($QR$) of the plane.

(b) *Any line parallel to* V can have that fact discovered by the parallelism of its plan to G.L.;

Fig. 173.

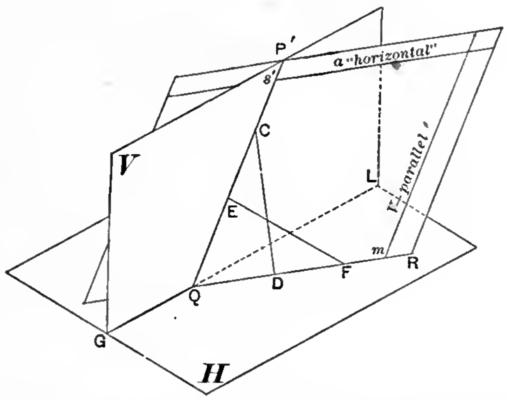
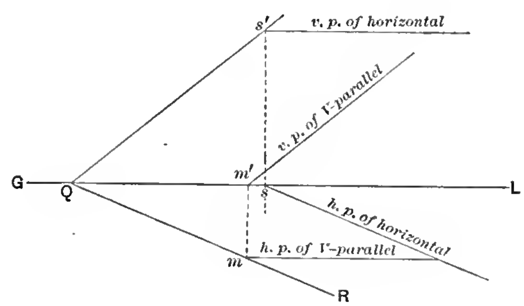



Fig. 174.




but when contained by a plane we shall call it a *V-parallel* of the plane. Such line will evidently be parallel to the v.t. of the plane in which it lies.

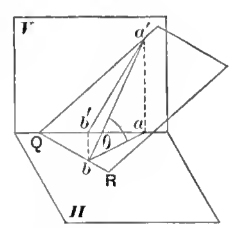
(c) The oblique lines $CD$ and $EF$, with those just described, illustrate the additional fact that *the traces of lines that lie in a plane will be found on the traces of the plane.* This furnishes the following—and usual—method of determining a plane, i.e., by means of two lines known to lie in it: Prolong the lines until they meet the plane of projection; their H-traces joined give the h.t. of the plane. Similarly for the vertical trace of the plane.

(d) Conversely, *to assume a line in a plane* assume its traces on those of the plane.

(e) Two intersecting lines or two parallel lines determine a plane. Three points not in a straight line, or a point and a straight line may be reduced to either of the foregoing cases.

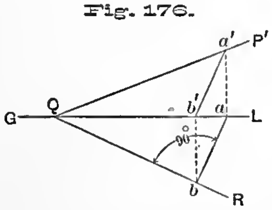
301. *Lines of declivity.* A line lying in a plane and making with H or V the same angle as the plane, is called a *line of declivity.*

Fig. 175.



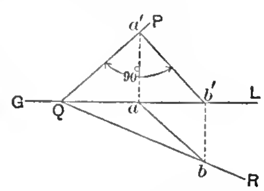

Figs. 175 and 176 show a line of declivity with respect to H. Both the line and its plan must be perpendicular to the h.t. of the plane; for the inclination of a line to its plan is that of the line to H; and if, at the same time, the measure of a dihedral angle, such lines must, by elementary geometry, be perpendicular to the intersection of the planes. Fig. 177 gives a line of declivity with respect to V.

Fig. 176.




302. *Limiting angles.* Were the plane $P'QR$ (Fig. 175) rotated on its line of declivity ($ab$, $a'b'$) it would make an increasing angle with H until perpendicular to it. If, then, a plane contain a line, its inclination-limits are 90° and that of the line.

Fig. 177.




On the other hand, the line of declivity might be turned *in the plane, until* horizontal. The limits of the inclinations of lines in a plane are therefore 0° and that of the plane.

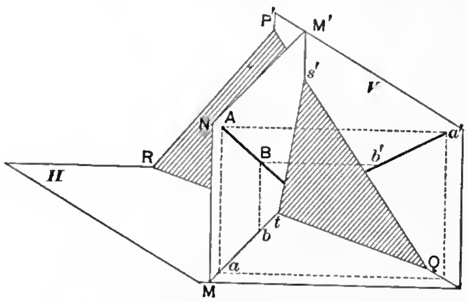
A plane may make 90° with H and be parallel to V, or vice versa; or it may be perpendicular to *both* H and V; the limits of the *sum* of its inclinations to H and V are thus 90° and 180°.

If parallel to G.L., but inclined to H and V, the sum of the inclinations of the plane is the lower limit, 90°.

If a plane is equally inclined to both H and V, but cuts G. L., its traces will make equal angles with the latter.

303. *Lines perpendicular to planes.* If a right line, as $AB$ (Fig. 178), is perpendicular to a plane $P'QR$, its projections will be perpendicular to the traces of the plane. For the plan of the line lies in the h. t. of its H-projecting plane; the latter plane is—from its definition—perpendicular to H; is also perpendicular to the given plane by virtue of containing the given line; hence is perpendicular to the h. t. of the given plane, since such h. t. is the intersection of the latter with H.

Fig. 178.



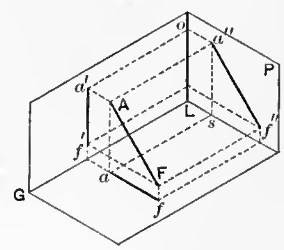

The same principles apply to the relation of the elevation of the line to the v. t. of the plane.

304. *Profile planes.* Any plane perpendicular to both H and V is called a *profile plane.* Such plane (P, Fig. 179) is used when *side* or *end* views of an object are to be projected. To bring V, H and P into one plane we suppose the latter first rotated into V about their line of intersection, $oL$, then both V and P about G. L. into H.

Projections on the profile plane are usually lettered with a double accent, the same as for any point revolved into or parallel to V.

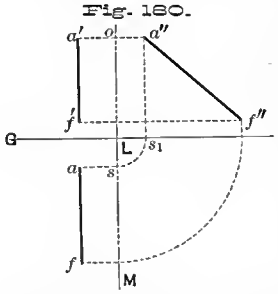
When, as in Fig. 179, all the projectors of a line $AF$ lie in the same projector-plane, both projections of the line will be perpendicular to G. L. The most convenient method of dealing with such line is to project it upon a profile plane and revolve the latter into V; or the profile projector-plane through the line might be directly revolved into V, carrying the line with it. The former method is illustrated in Figs. 179 and 180.

Fig. 179.




In this, as in many other constructions, we make use of the following principles:

Fig. 180.



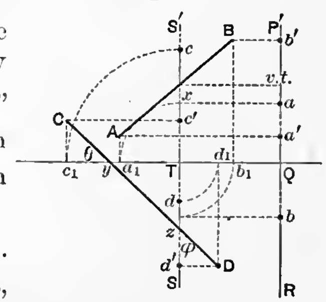

(a) All projections of one point on two or more *vertical* planes will be at the same height above H.

(b) If rotation occur about an axis that is perpendicular to H, each arc, described about that axis by a point revolved, will be projected on H as an equal circular arc; similarly as to V, if the axis is perpendicular to it.

Since in Figs. 179 and 180 we rotate upon a *vertical* axis, a projector, as $Aa''$, will be seen in $as$, drawn through the plan of $A$ and perpendicular to the h. t. of P. From $s$ a circular arc, $ss_1$ (Fig. 180), from centre $L$, will be the plan of the arc described by $a''$ of Fig. 179. From $s_1$ a vertical to the level of $a'$ gives $a''$. Similarly the projection $f''$ is obtained, which, joined with $a''$, gives $a''f''$, the profile view of the line $AF$.

305. As far as our view of what is in the first angle is concerned, the rotation just described amounts, practically, to the turning of H and V through an angle of 90°, so that instead of facing V we see it "edgewise," as a line $Mo$; while H appears also as a line, $GLs_1$. We thus get an "end view" into the angle. All figures lying in profile planes are then seen in their true form.

Fig. 181.

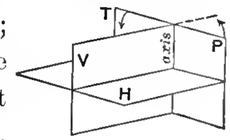



In Fig. 181 let us start with the entire system of angles thus turned. The ground line appears as a point, $T$; H and V as lines; and two lines, $AB$ and $CD$—each of which lies in a profile plane—are shown in their true length and inclination.

Perpendiculars to H from $A, B, C$ and $D$, give their revolved plans $a_1, b_1, c_1$ and $d_1$. V-projectors give $d', c'$, etc., the heights of the elevations of the points.

Revolving the whole system into its usual position, remembering, meanwhile, that the profile plane, P, turns on a *vertical axis* (as in Fig. 182) which divides P into parts which are on opposite sides of the axis both before and after revolution, we find $c_1$ at $c$; $d_1$ at $d$; $a_1$ at $x$. Assuming $SS'$ as the plane of the line $CD$, and that the plane of $AB$ is $RP'$, at a given distance $TQ$ from $SS'$, we find $a'$ and $b'$ on $RP'$ at the same level as $A$ and $B$; while $a$ is derived from $x$, and $b$ from $b_1$ as shown. The elevations $c'$ and $d'$ on $SS'$ are on the level of $C$ and $D$ respectively.

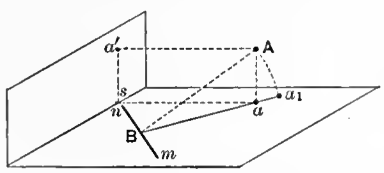
Fig. 182.




306. *Rabatment, and analogous rotations.* The term *rabatment*, already employed (Art. 292) to indicate the rotation of a *plane* about one of its traces until it comes into a plane of projection, is also used to denote the rotation of a *point* or *line* into H or V about an axis in such plane.

Restoration to an original space-position will be called *counter-rabatment* or *counter-revolution.*

Fig. 183.

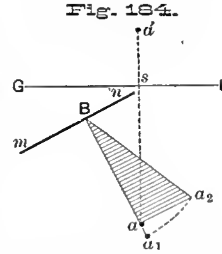



In Fig. 183 we have $a_1$ as the rabatment of $A$ into H, about an axis $mn$. $Ba_1$ is equal to $BA$, i.e., to the actual distance of $A$ from the axis, and which is evidently the hypothenuse of the triangle $ABa$, whose altitude is the H-projector $(Aa)$ of the point, and whose base is the h.p. $(aB)$ of the real distance, $AB$.

Were the axis *parallel to* and not *in* H we would state the principle thus: In revolving a point about, and to the same level with, an *horizontal* axis it will be found on a perpendicular drawn through the h.p. of the point to the h.p. of the axis, and at a distance from the latter equal to the hypothenuse of a right-angled triangle whose altitude equals the *difference of level* of point and axis, and whose base is the h.p. of the real distance. Were the axis *in* or *parallel to* V, the *base* of the triangle constructed would be the *v.p.* of the desired distance, and the altitude would be—in the first case—the V-projector of the point, and—in the second case—the difference of distances of point and axis from V. *In any case*, the vertex of the right angle, in the triangle constructed, is the projection of the original point on that plane of projection *in which* or *parallel to which* the axis is taken.

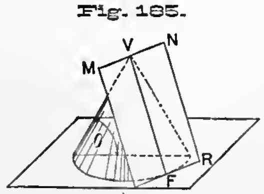
*In usual position* the foregoing construction would appear thus: With the point given by its projections, $(aa'$, Fig. 184), let fall a perpendicular through $a$ to $mn$; prolong this indefinitely; make $a$ the vertex of the right angle in a right triangle of base $Ba$, altitude $a's$; then the hypothenuse of such triangle, used as a radius of arc $a_2a_1$ (centre $B$) gives $a_1$ as the revolved position desired.

Fig. 184.




307. In applying the foregoing principles in the following problems we shall frequently find it convenient to employ the right cone as an auxiliary surface. All the elements of such a cone are equally inclined to its base, and a tangent plane to the cone makes with the base the same angle as the elements. The element of tangency is a line of declivity of the plane with respect to the base of the cone.

Fig. 185.




If the base of the cone is on H, the h.t. of a tangent plane will be tangent to the base of the cone; similarly for its v.t. were the base in V.

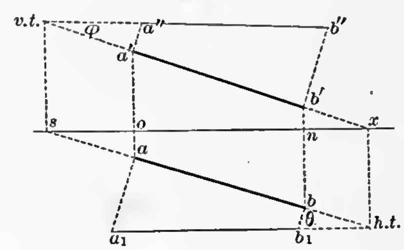
308. *Prob. 1. From the projections of a line to determine* (a) *its traces;* (b) *its actual length;* (c) *its inclinations, $\theta$ and $\phi$, to* H *and* V *respectively.*

(a) *The traces of the given line*, when it is oblique to both H and V, as in Fig. 186, are found by the rule given in Art. 293.

(b) *The actual length* of a line may be found either (1) by *rabatment into* H or V, or (2) *by rotation until parallel to* H or V.

*By the first method* rabat the line on its plan $a\,b$ as an axis. It will show the true length on H in $a_1\,b_1$, the distance $a\,a_1$ equalling $a'\,o$—the original height of the point $A$ above H; similarly, $b\,b_1$ equals $b'\,n$. Notice that $a_1\,a$ and $b_1\,b$ must be perpendicular to the axis, and that each is the projection of a circular arc, described by the point revolved.

Fig. 186.

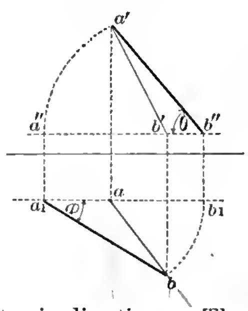


The point where a revolved line meets the axis of rotation is common to both the original and the revolved positions of the line. In illustration see h. t. and v. t., Fig. 186.

If we make $a'\,a''$ and $b'\,b''$ perpendicular to $a'\,b'$ and equal to $a\,o$ and $b\,n$ respectively, we have in $a''\,b''$ the real length, shown on V.

*By rotation till parallel to a plane of projection*, as H, either extremity of the line may be brought to the level of the other, when the new plan will show the actual length. Thus, (Fig. 187), *using a horizontal axis*, (the V-projector of $b\,b'$) $a'$ may be brought to $a''$, at the level of $b'$, by an arc, centre $b'$, radius $a'\,b'$. The circular arc $a'\,a''$ thus described has its h. p. in $a_1\,a$, the distance from V having been constant during the rotation, since the axis was perpendicular to V. In $a_1\,b$ we then have the real length sought.

Fig. 187.



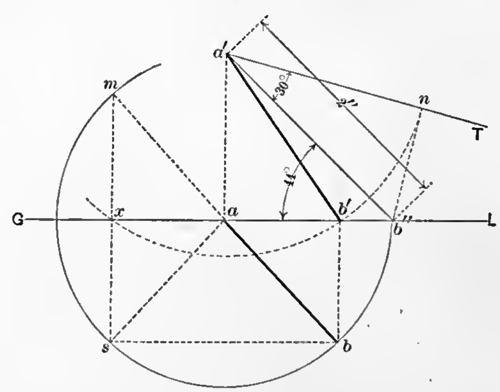
*If we rotate the line on a vertical axis* through $a$ until $b$ reaches $b_1$, we will find the v. p. of the revolved point at $b''$, on its former level. The new projection, $a'\,b''$, is again the real length, now projected on V.

(c) *The inclinations, $\theta$ and $\phi$, to* H *and* V *respectively.* Either of the foregoing constructions for showing the real length of a line solves also the problem as to inclination. Thus, in Fig. 186, the rabatted lines make with their axes of rabatment the angles sought. In Fig. 187 we have $a'\,b''$ inclined $\theta°$ to H, while $b\,a_1$ makes with $a\,a_1$ the angle $\phi$.

*When the line lies in a profile plane* the traces, length and inclination are found by means of the operation described in Art. 305 and illustrated by Fig. 181. In that figure, were $c\,d$ and $c'\,d'$ given, we would carry $c$ and $d$ about $T$ as a centre to $c_1$ and $d_1$, whence perpendiculars to their former levels would give $C$ and $D$; joining the latter we would have $C\,D$, whose v. t. is at $z$; h. t. (not shown) at a distance $T\,y$ above $T$; while $\theta$ and $\phi$ are seen in actual size at $y$ and $z$.

309. *Prob. 2. To determine the projections of a line of given length, having given its angles, $\theta$ and $\phi$, with* H *and* V *respectively.* If with the line we generate a vertical right cone of base angle $\theta$, four elements could be found on the cone, each of which would make with V an angle equal to $\phi$ and therefore fulfill all the conditions.

Fig. 188.



The sum of $\theta$ and $\phi$ can obviously not exceed 90°, and when equalling that limit there could be but two solutions on a given cone, and both would lie in a profile plane.

For data take *length of line*, 2″; $\theta = 44°$; $\phi = 30°$. From any point $b''$, on G. L., draw *in* V a line $b''\,a'$, of the given length and at the angle $\theta = 44°$ with G. L. The plan of this line is $a\,b''$, which use as the radius of the base of a semi-cone of vertical axis $a\,a'$.

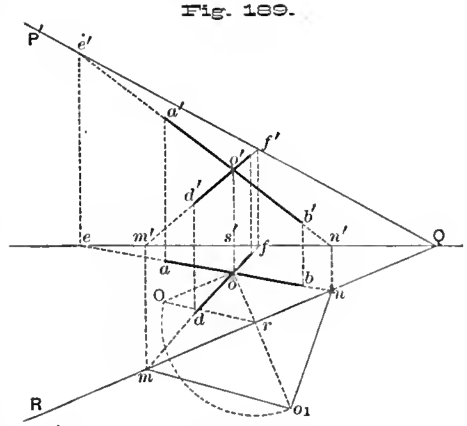
Remembering that the inclination of a line determines the length of its projection, we next ascertain how long the projection of a two-inch line will be when inclined 30° to a plane. Draw-

ing $a'T$ at 30° to $a'b''$, and projecting $b''$ perpendicularly upon it at $n$, we find $a'n$ as the invariable length sought. Arc $nb'$, from centre $a'$, gives $a'b'$ as the v. p. of an element of the cone, whence $ab$ follows, as the plan of the desired line.

As arc $nb'$, continued, would cut G. L. at $x$, whence $s$ and $m$ could be derived, we would find $sa$ and $ma$ as the plans of two more elements fulfilling the conditions. Also, in line with $bb'$, one more point (omitted to avoid confusing the solutions) could be found, on the rear of the cone.

310. *Prob. 3. To determine the plane containing* (a) *two intersecting lines;* (b) *two parallel lines.* The lines $CD$ and $EF$ of Fig. 173 illustrate, pictorially, the principle of the construction.

Fig. 189.




(a) In Fig. 189 let $oo'$ be the point of intersection of the lines $AB$ and $CD$. Prolong the lines and obtain their traces, as in Art. 293. $RQ$, the h. t. of the plane, is the line connecting $m$ and $n$, the H-traces of the lines. Similarly, $P'Q$ passes through the V-traces, $e'$ and $f'$.

(b) *Parallel lines* have parallel projections, and the h. t. or v. t. of their plane may be obtained by joining the like traces of the lines. (Art. 300).

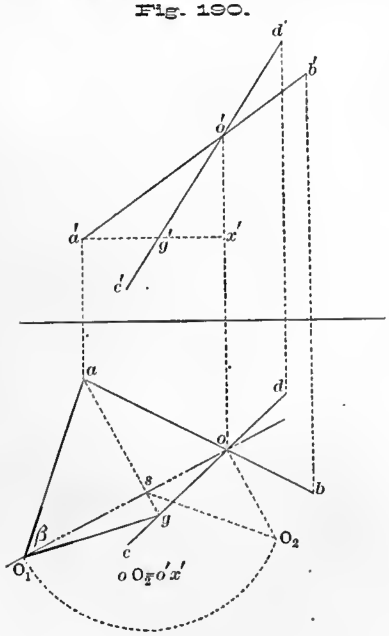
311. *Prob. 4. To show in true size the angle between two lines.*

(a) *If the actual angle is* 90° it will be projected as such when *one* or *both* of its sides are parallel to the plane of projection: for, if either is parallel, the traces of the projecting planes—in which lie the projections of the lines—will evidently be perpendicular to each other; if either side of the angle in space be then rotated about the other side as an axis, it will turn in its previous projector-plane, and its projection will still fall upon the same trace as before.

(b) *In general, to show the true size of any angle,* rotate its plane either *into* or *parallel to* H or V.

In Fig. 189 the angle whose vertex is $oo'$ is shown by obtaining the plane $RQP'$, of its sides, then rabatting about $RQ$ into H. The vertex reaches $o_1$ after describing an arc whose plane is perpendicular to $RQ$ and which is projected in $oo_1$. The actual space-distance of O from $r$—the point on the axis, about which it turns—is the hypothenuse of a right triangle whose altitude $oO = o's'$, and whose base is $or$. Joining $o_1$ with $m$ and $n$—the intersections of the axis by the lines revolved—gives $mo_1n$, the angle sought.

Fig. 190.



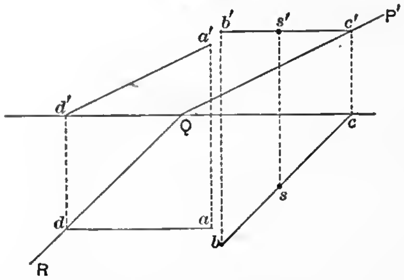

*To obtain the angle without having the traces of its plane* we may use as an axis the line connecting points—one on each line—and equidistant from a plane of projection. Thus, in Fig. 190, we find $a'$ and $g'$ on the same level; then $ag$ is the plan of the axis, about which $o$ rotates to $O_1$, giving $oO_1g$ for the desired angle.

312. *Prob. 5. To draw a horizontal and a V-parallel in a plane.*

As shown in Figs. 173 and 174, any horizontal line has a v. p. parallel to G. L., and—if contained by a plane—must be parallel to the h. t. of the plane, and also meet V on its v. t.; hence any line $b'c'$, (Fig. 191) parallel to G. L., will do for the v. p. of the desired line; the intersection of $P'Q$ by $b'c'$ will be the v. t. of the line, a vertical through which to G. L. gives $c$—one point of the plan $cb$, whose known direction (parallel to $RQ$) enables it to be immediately drawn.

A V-parallel is parallel to the v. t. of the plane in which it lies, and meets H on the h. t. of the plane; hence on $RQ$ assume $d$ as the h. t. of the desired line, project to $d'$ and draw $d'a'$ parallel to $P'Q$; then $da$, parallel to G. L., represents (with $d'a'$) a V-parallel of the plane.

Fig. 191.




Additional conditions might be assigned to either kind of line, as, for example, the distance from the plane to which the line is parallel; the quadrant; the length of the line, or that it should contain a certain point of the plane.

313. *Prob. 6. Having one projection of a point in a plane, to locate the other projection.* If a point is on a line its projections will be on the projections of the line, and also on a perpendicular to G. L.; if, therefore, the plan $s$ (Fig. 191) of the point is given, we may draw either a horizontal or a V-parallel through the point and in the plane. Choosing the former we have $bsc$, parallel to $RQ$. From $c$, where it meets G. L., a vertical line cutting $P'Q$ gives its v. t., through which draw $c'b'$ parallel to G. L. The desired v. p. is then $s'$, on $c'b'$ and vertically above $s$.

Were the *elevation* of the point given we would find its plan on the h. p. of a V-parallel drawn through the point and in the plane; or we might start with the v. p. of a horizontal.

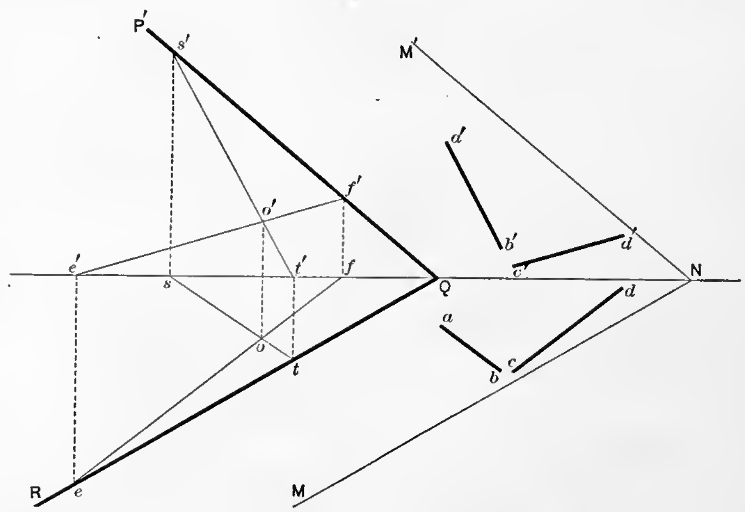
314. *Prob. 7. To pass a plane through three points not in the same straight line. Any two* of the three lines that would connect the points by pairs would determine the plane by the first case of Prob. 3; while the line joining any two of the points, together with a parallel to it through the third point, becomes the second case of the same problem.

315. *Prob. 8. To pass a plane through one line and parallel to another* draw through any point of the first line a parallel to the second; such parallel will, with the first line, determine the plane.

316. *Prob. 9. To pass a plane through a given point and parallel to a given plane.*

(a) Two lines through the given point and parallel to any pair of lines in the given plane would determine a plane fulfilling the conditions.

Fig. 192.




In Fig. 192, with $oo'$ as the given point, and $M'NM$ as the given plane, assume *in the latter* any two lines, $AB$ and $CD$; parallel to these lines and through $oo'$ draw $ST$ and $EF$, whose traces will determine those of the plane sought. (The student should note that $ab$, $a'b'$, and $cd$, $c'd'$, are not random projections, but fulfill the condition of Case (c), Art. 300).

Since parallel planes have parallel traces, *one* line through $oo'$ would suffice. For example, $st$, $s't'$, parallel to $ab$, $a'b'$, gives the traces $t$ and $s'$, through which draw $RQ$ and $QP'$, parallel respectively to the like traces of the given plane.

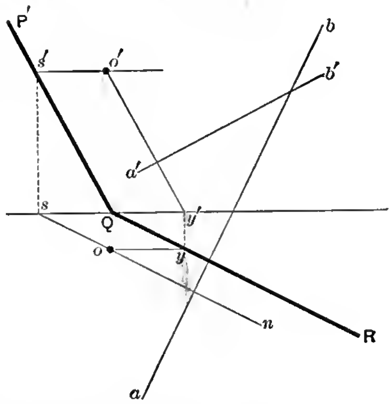
(b) Knowing the direction of the required traces we may determine the plane by means of either a horizontal or a V-parallel through the point.

In Fig. 191, were $ss'$ the given point, and had a plane been given whose traces were parallel

to those of $P'QR$, the latter would be the result of applying the method by a horizontal ($bc$, $b'c'$) through the point.

317. *Prob. 10. To pass a plane through a given point and perpendicular to a given line.* Since, by Art. 303, the traces of the desired plane will be perpendicular to like projections of the line, draw through the given point, $oo'$, Fig. 193, either a *horizontal* or a V-*parallel* of the desired plane; the trace of either line, and the known directions of the required traces, suffice to solve the problem.

Fig. 193.

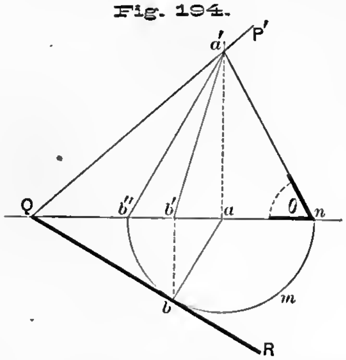


The plan, $on$, of a *horizontal*, will be perpendicular to $ab$—the plan of the given line; through $s'$, the v.t. of the horizontal, we draw $P'Q$ perpendicular to $a'b'$, for the v.t. of the plane; then $QR$ perpendicular to $ab$ for the desired h.t.

For the V-*parallel* through $oo'$ draw $o'y'$ perpendicular to $a'b'$, and $oy$ parallel to G.L., then through the trace $y$ a line at $90°$ to $ab$ will be the h.t. of the plane sought, whence $QP'$ follows, at the same angle to $a'b'$.

318. *Prob. 11. To determine* (a) *the angles $\theta$ and $\phi$ made with* H *and* V *respectively, by a given plane;* (b) *the angle between the traces of the plane.* From the properties of the cone and its tangent plane mentioned in Art. 307, we may solve the problem by generating a cone with a line of declivity of the plane, and ascertaining the inclination of such line.

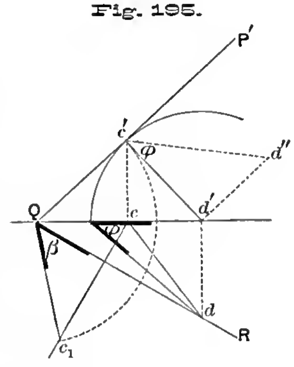
Fig. 194.



(a) In Fig. 194 let $P'QR$ be the plane. The projections $a'b'$ and $ab$—the latter perpendicular to $RQ$—represent a line of declivity of the plane with respect to H. With it, and about $a'a$ as an axis, generate a semi-cone. When the generatrix reaches V, either at $b''a'$ or $a'n$, its inclination to G.L. shows the base angle $\theta$ of the cone, and therefore the inclination of the given plane.

With respect to V the construction is analogous. A line of declivity with respect to V has its v.p. ($c'd'$) at $90°$ to $P'Q$, (Fig. 195). Using $d$, on the h.t. of the plane, as the vertex of a semi-cone of horizontal axis $dd'$, we find the base of the cone tangent to $P'Q$ at $c'$. Carrying $c'$ to the ground line, about $d'$ as a centre, and joining it with $d$ gives the angle $\phi$ sought.

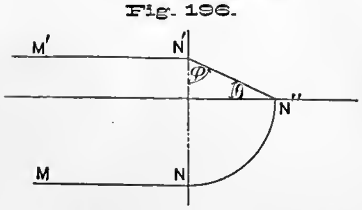
Fig. 195.



This problem might also be readily solved by rabatting the line of declivity into a plane of projection. Thus making $d'd''$ perpendicular to $c'd'$ and equal to $d'd$, we find the angle $\phi$ between $c'd'$ and $c'd''$.

*For a plane parallel to* G.L. use the auxiliary *profile* plane, rotating its line of intersection with the given plane as in Fig. 196.

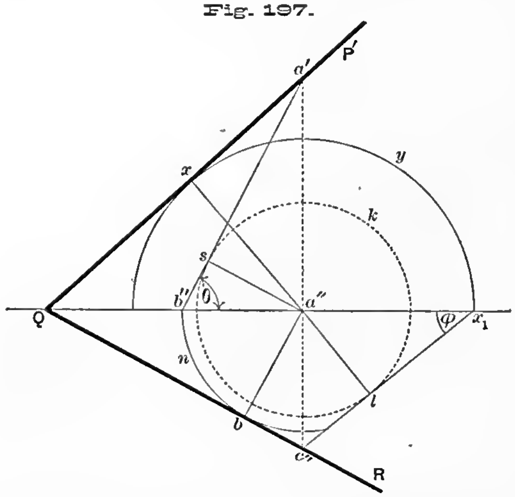
Fig. 196.



(b) *The angle between the traces* is obtained by rabatment of the given plane about either of its traces. In Fig. 195, using trace $RQ$ as an axis and rotating $QP'$ about it, any point, $c'$, thus turned will describe an arc projected in a perpendicular through $c$ to $QR$. $Q$ being on the axis is constant during this rotation, and the distance from it to $c'$ will be the same after as before revolution; therefore cut $cc_1$ by an arc, centre $Q$, radius $Qc'$; the desired angle is $\beta$ or $c_1QR$.

319. *Prob. 12. To obtain the traces of a plane, having given its inclinations,* $\theta$ *and* $\phi$, *to* H *and* V *respectively.* This is, obviously, the converse of Prob. 11 and is, practically, the same construction in reverse. The required plane will be tangent, simultaneously, to two semi-cones of base angles $\theta$ and $\phi$, and having axes (a) in V and H respectively, and (b) in the same *profile plane.*

Fig. 197.

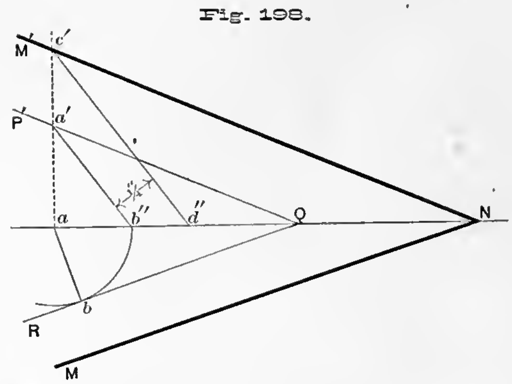



Assume in V any vertical line $a'a''$ as the axis of the $\theta$-cone, and draw from any point of it, as $a'$, a line $a'b''$, at $\theta°$ to G. L.; use $a''b''$, the plan of this line, as radius of the base of the vertical semi-cone, to which the desired h. t. of the required plane will be tangent. The line $a''s$ shows the perpendicular distance from $a$ to the point of intersection of the two elements of tangency of the required plane with the cones; hence the generatrix of the $\phi$-cone must, when in H, be at an angle $\phi$ with G. L., and tangent to arc $skl$ of radius $a''s$, centre $a''$, and is therefore $x_1c''$. Draw $x_1yx$ for the half-base of the $\phi$-cone. The h. t. of the plane sought is then $RQ$, drawn through $c''$ and tangent to the base of the $\theta$-cone; while the v. t. is $QP'$, tangent to base $x_1yx$.

For the limits of $\theta + \phi$ refer to Art. 302.

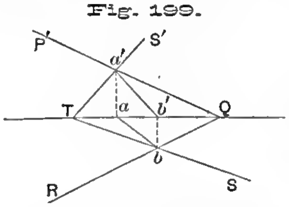
320. *Prob. 13. To draw two parallel planes at a given distance apart.* Parallel planes have parallel traces. To draw (Fig. 198) a plane at a distance of $\frac{1}{4}''$ from plane $P'QR$, rotate a line of declivity of $P'QR$ into V at $a'b''$. Draw $c'd''$, parallel to and $\frac{1}{4}''$ from $a'b''$, to represent (in V) the line of declivity of the plane sought. It meets $aa'$ (prolonged) at $c'$, through which draw $c'N$ parallel to $P'Q$. From $N$ draw the trace $MN$, parallel to $RQ$. $M'NM$ then fulfills the conditions.

Fig. 198.




321. *Prob. 14. To obtain the line of intersection of two planes.* As two points determine a line, we have merely to find two points, each of which lies in both planes, and join them. In Fig. 199 the line *in space* which would join $a'$, the intersection of the vertical traces of the planes, with $b$, the corresponding point on the horizontal traces, would be the required line. Its projections are $ab$, $a'b'$.

Fig. 199.

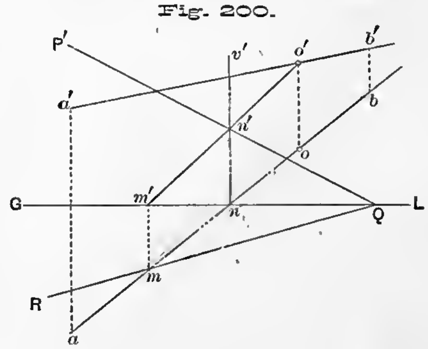



Were the H-traces of the planes parallel, the line of intersection would be horizontal. Were the V-traces parallel, their line of intersection would be a V-parallel of each plane.

If both planes were parallel to G. L., a profile plane might advantageously be employed in the solution. The line sought would be parallel to G. L.

322. *Prob. 15. To find the point of intersection of a line and plane* find the line of intersection of the given plane by *any* auxiliary plane containing the line; the given line will meet such line of intersection in the desired point.

Fig. 200.




In Fig. 200 $mnv'$ is an auxiliary *vertical* plane through the given line $ab$, $a'b'$. The given

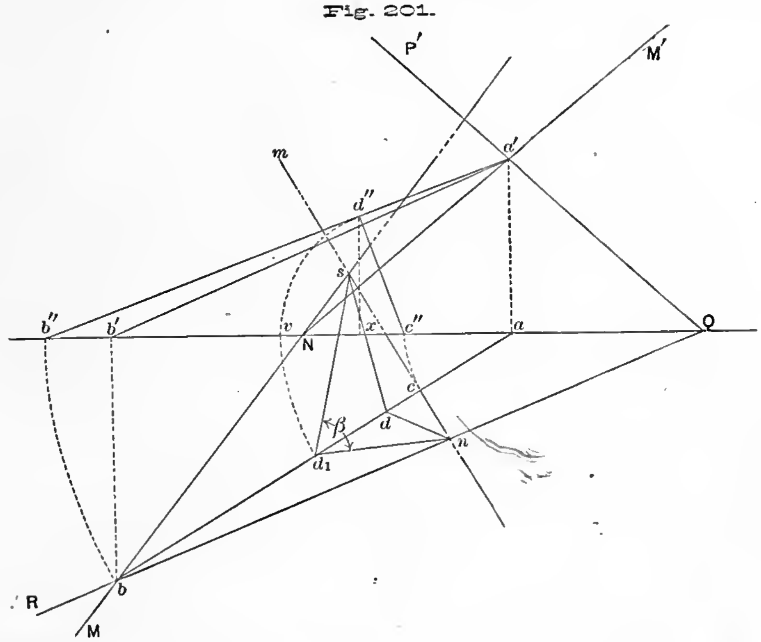
plane, $P'QR$, intersects the plane $mnv'$ in the line $mn$, $m'n'$. The elevations $a'b'$ and $m'n'$ meet at $o'$—the v. p. of the point sought, whose h. p. must then be on a vertical through $o'$ and on both the other projections of the lines, hence at $o$.

323. *Prob. 16. To show the actual size of the angle between two intersecting planes.*

A plane perpendicular to the line of intersection of the planes will cut from them lines whose plane angle equals the dihedral angle sought.

Let $a'b'$, $ab$ be the line of intersection of the planes $P'QR$ and $M'NM$. Any line $mn$, drawn perpendicular to $ab$, may be taken as the h.t. of an auxiliary plane perpendicular to the line of intersection. The part $sn$, included between the H-traces of the given planes, will—with the lines cut from the latter—form a triangle whose vertical angle is that desired. The altitude of this triangle is a line projected on $cb$, and—in space—is a perpendicular from $c$ to $AB$. To ascertain its length rabat the H-projecting plane of $AB$ into V, about $a'a$ as an axis. The point $c$ appears at $c''$, and the line $AB$ at $a'b''$. The altitude sought appears in actual size at $c''d''$, perpendicular to $a'b'$. From $c$ lay off on $cb$ the length $cd_1$ equal to $c''d''$; $sd_1n$, or $\beta$, is the required angle.

Fig. 201.




A vertical from $d''$ to G. L., thence an arc to $cb$ (from centre $a$), gives $d$, the plan of the vertex of the angle in space.

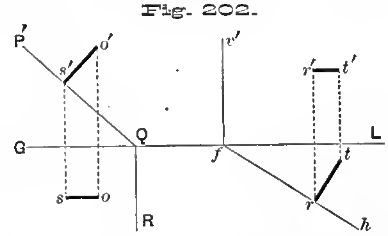
When both planes are parallel to G. L. use an auxiliary profile plane in solving.

324. *Prob. 17. To determine the angle between a line and a plane* let fall a perpendicular from any point of the line to the plane; the angle between the perpendicular and the given line will be the complement of the desired angle. The principles involved are those of Arts. 303 and 311.

325. *Prob. 18. To show in its true length the distance from a point to a plane.*

*If the plane is perpendicular to* H *or* V the distance is evident without any construction. Thus, in Fig. 202, a perpendicular through the point to the plane is seen in its true length, $o's'$, because in space it is parallel to V.

Fig. 202.



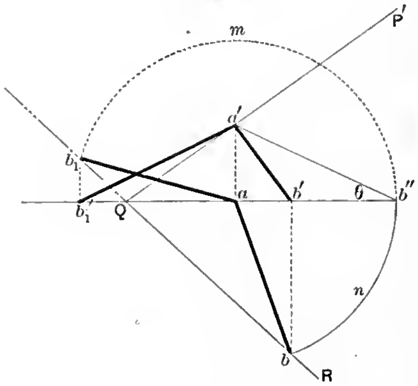

Similarly, $tr$ is the actual length of the perpendicular from $tt'$ to the vertical plane $v'fh$.

*When the plane is oblique to both* H *and* V let fall (by Art. 303) a perpendicular from the point to the plane; find by Prob. 15 the intersection of such perpendicular with the plane, and show, by Prob. 1 (b), the real length of the distance between the original point and the one thus obtained.

326. *Prob. 19. To draw a line of given inclination in a plane.* We have seen, in Art. 302, that we can assign to the line no greater inclination than that of the plane. Starting with the line *in* V (Fig. 203), make its intersection, $a'$, with $P'Q$, the vertex of a cone whose base angle $\theta$ is the inclination assigned to the line. With $ab''$ as a radius describe the base of this cone. This cuts

$RQ$—the h.t of the plane—in $b$ and $b_1$, each of which, joined in space to $a'$, would give a line fulfilling the conditions. In projection the two solutions are $a'b'$, $ab$, and $a'b'_1$, $ab_1$.

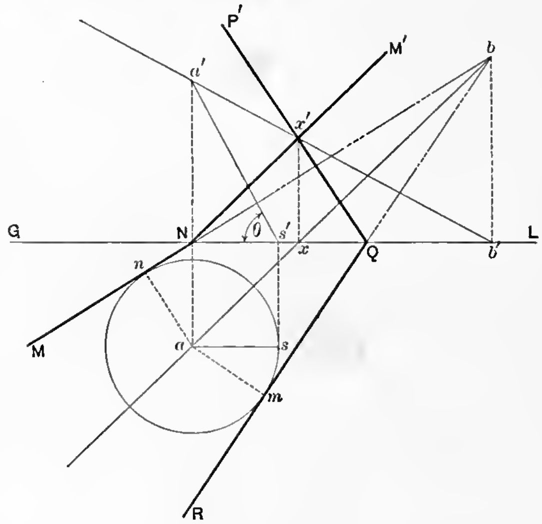
Fig. 203.




*If the line is to pass through a given point of the plane* make that point the vertex of the cone employed in the construction, taking the initial position of the generatrix parallel to V.

327. *Prob. 20. To obtain the traces of a plane containing a given line and making a given angle with* H *or* V.

Fig. 204.




Make any point on the line the vertex of a cone whose base angle is the assigned inclination; the required plane will be tangent to such cone, and its traces will pass through the like traces of the given line.

In Fig. 204 let $ab$, $a'b'$ be the given line. On it take any point $aa'$ as the vertex of a cone generated by a line $a's'$, $as$, inclined $\theta°$ to H, and, at first, parallel to V. Since the H-traces of the two possible planes fulfilling the conditions would be tangent to the like trace of the cone, draw from $b$—the h.t. of the given line—the tangents $bR$ and $bM$ to the circle $msn$. These cut G.L. at $Q$ and $N$, each of which is then joined with $x'$—the v.t. of the given line—to obtain the v.t. of one of the desired planes.

328. *Prob. 21. To determine the projections of two intersecting lines, having given their inclinations to* H *or* V, *and their angle with each other.*

Fig. 205.

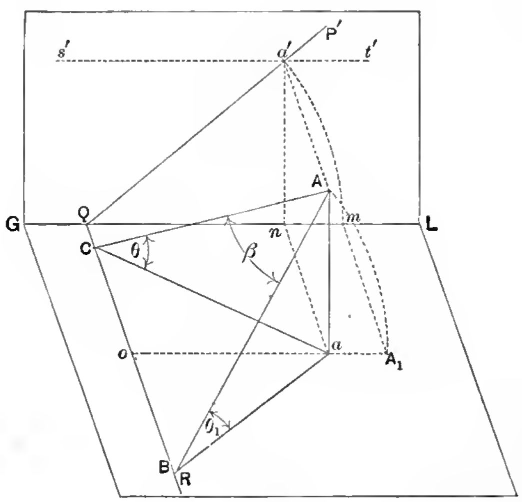
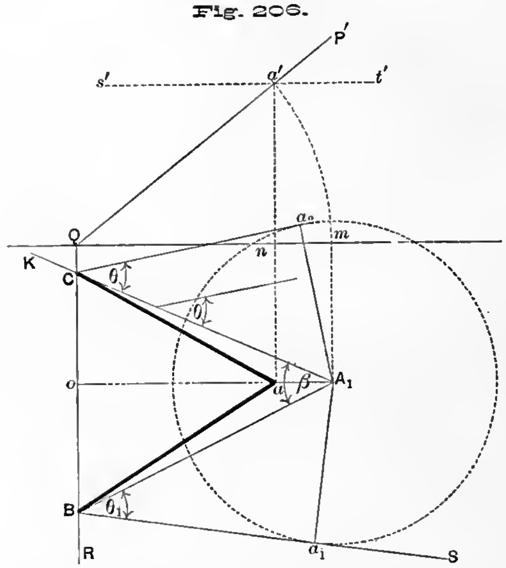



Fig. 206.




It will be seen from Fig. 205 that, in space, either line $AB$, $AC$, will be the hypothenuse of a right-angled triangle whose plane is perpendicular to that plane of projection relative to which the

inclinations are given; that the altitudes (perpendiculars) of these triangles will not only be equal but coincide (as in $A a$), and that the angles $\theta$ and $\theta_1$, at their bases, will be the given inclinations to the plane of projection.

Let the two lines, $A C$ and $A B$, be inclined $\theta°$ and $\theta_1°$ respectively to H, and also make with each other some angle $\beta$; to find their projections. From any point on H, as $A_1$ (Fig. 206), draw two lines, $A_1 B$ and $A_1 K$, making the angle $\beta$ with each other. At *any* point on $A_1 B$, as $B$, draw a line $B S$, making with $A_1 B$ an angle equal to the inclination to H of the line $A B$ in space. From $A_1$ draw $A_1 a_1$ perpendicular to $B S$. With $A_1 a_1$ as a radius, and $A_1$ as a centre, describe a circle. Tangent to this circle draw a line meeting $A_1 K$ at $C$, at an angle equal to the other given inclination, $\theta°$. $B$ and $C$ are then the H-traces of the lines $A B$ and $A C$, and the line $B C$ will form part of the h. t. of their plane. For convenience take the vertical plane of projection perpendicular to the plane of the lines, so that the triangle $A B C$ will be projected upon it in a straight line. This condition will be met by taking G. L. perpendicular to $B C$. On V draw $s't'$ parallel to G. L., and at a distance from it equal to $A_1 a_1$ or $A_1 a_2$, either of which represents the actual height of the point $A$ in space, when the lines $A B$ and $A C$ make the given angles with H. With $Q$ as a centre and radius $Q m$ ($m$ is the v. p. of $A_1$) describe an arc, $m a'$, intersecting $s't'$ at $a'$. This arc is the v. p. of the arc that $A_1$ describes about $R Q$ in reaching its space-position. $A_1 o$ is the h. t. of the plane in which $A_1$ rotates. The h. p. of $A$ is then at $a$—the intersection of $o A_1$ by $a n a$, the projector plane of $A$. The desired plans of the given lines are $a B$ and $a C$, while $a' Q$ is their common elevation.

*Limits of the conditions that may be assigned.* The sum of the given inclinations and the angle between the lines will reach its *maximum*, 180°, when the lines lie in a profile plane.

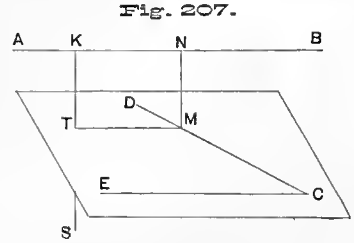
The length of but one of these lines may be predetermined. If assigned, such length must be made the hypothenuse of the first triangle constructed. The longer line will, obviously, make the smaller angle.

329. *Prob. 22. To determine the traces of two mutually perpendicular planes, having given their inclinations to* H *or* V. Two planes will be mutually perpendicular if either contains a line that is perpendicular to the other; to solve this problem we therefore determine, by the first step of Prob. 11, the traces of a plane at one of the given inclinations, $\theta$, to H; by Art. 303 erect a perpendicular to this plane; and lastly, by Art. 327, obtain the traces of a plane containing such perpendicular and tangent to a cone (a) whose vertex is on the perpendicular, and (b) whose base angle is the other assigned inclination.

The planes will intersect in a *horizontal* line when the sum of their dihedral angle and their inclinations to H has its minimum value, 180°.

330. *Prob. 23. To determine the common perpendicular to two lines not lying in the same plane.* Only a pictorial representation of the various steps is shown (Fig. 207), their orthographic counterparts having been already fully described in detail. Let $A B$ and $C D$ be the given lines. Pass a plane through either line parallel to the other. The parallelogram represents such a plane, determined by $C D$ and $C E$, the latter a parallel to $A B$ and drawn through any point of $D C$. By Art. 303 drop a perpendicular $K T S$, from *any* point of $A B$ to the auxiliary plane, and obtain its trace $T$ on the latter by Art. 322. $T M$, drawn parallel to $A B$, will be the projection of the latter on the plane. $M N$, a parallel to $T K$ through the intersection of $T M$ and $D C$, will be the desired perpendicular, whose true length may be ascertained as in Art. 308.

Fig. 207.



### DEFINITIONS AND VARIOUS CLASSIFICATIONS OF THE MORE IMPORTANT LINES AND SURFACES.

In our graphical constructions the draughtsman is concerned—not with the *contents* of solids—but with the *form* and *relative position* of the *surfaces* that bound them, the *lines* in which such surfaces meet, and the *points* which are the intersections or extremities of such lines. The relations of these space-elements to their orthographic projections having been established in Arts. 228–330, it seems advisable, before solving special problems involving them, to present a sort of bird's-eye glimpse into the province of their application, since such course will not only enlarge the student's horizon as to the extent of the mathematico-graphical field, but also put the main facts in convenient form for reference and review; the remainder of the chapter is, therefore, devoted to definition, classification, illustration, and some of the important properties of the principal lines and surfaces with which the mathematician and draughtsman have to deal, together with a summary of the principles of construction later employed.

Since, unfortunately, line projections usually give scarcely a hint of the beauty of space-forms, shaded views in oblique projection are presented in the interest of attractiveness.

#### ALGEBRAIC AND TRANSCENDENTAL LINES AND SURFACES.—DEGREE.—ORDER.—CLASS.

331. *Algebraic.—Transcendental.* Taking up first the distinctions of Analytical Geometry (for definition see page 4) we find that a line or surface is *algebraic* if the relations of the co-ordinates of its points can be expressed by an equation which may be reduced to a finite number of terms, involving positive, integral powers of the variables. If, however, its equation involves other functions—as trigonometrical, circular, exponential, logarithmic—the curve or surface is called *transcendental.*

332. *Degree.—Order.* The *degree* of an algebraic curve or surface is that of the equation expressing it. The term *order* is used synonymously with *degree.*

The degree, or order, of an algebraic curve is the same as the maximum number of points in which it can be cut by a plane, and for a plane curve is the number of times it can be met by a right line in its plane.

*The straight line* is the only line of the *first order.*

*The conic sections* are the only mathematical curves of the *second order.*

Of the other curves treated in Chapter V the Witch of Agnesi and the Cissoid are of the *third* order; the Limaçon, Cardioid, Trisectrix, Cartesian ovals and Cassian ovals are of the *fourth;* and the remainder—Helix, Sinusoid, the Trochoids in general, Catenary, Tractrix, Involute and the Spirals—are *transcendental.*

Since transcendental equations expand into an infinite series their degree is assumed infinite, from which we conclude that a straight line may have an infinite number of intersections with a transcendental plane curve; analogously for a plane and a transcendental space curve.

333. If all plane sections of a surface are of a certain order the surface is of the same order.

A *plane* is the only surface of the *first* order.

*Conicoids* or *quadrics* (see Art. 367) are *all* and *the only* surfaces of the *second* order.

Of the other surfaces defined in the following articles the Conoid of Plücker (Cylindroid of Cayley) is of the *third* order; the Cyclide of Dupin, Cono-cuneus of Wallis, the Cylindroid of Frézier, Corne de Vache, Conchoidal Hyperboloid of Catalan and the Torus are of the *fourth;* and the others are *transcendental.*

334. *Class.* The *class* of a *plane curve* is the number of tangents that can be drawn to it from a point in its plane.

The *class* of a *non-plane curve* is the number of osculating planes (Art. 380) that can be passed to it through any point in space.

The *class* of an algebraic *surface* equals the number of possible tangent planes to it through any line in space.

LINES AND SURFACES AS ENVELOPES.—GENERATION BY CONTINUOUS MOTION.—REVOLUTION.—TRANSPOSITION.

335. *Envelopes.* If the points of a line are obtained as the consecutive intersections of the curves of a series, the line would be tangent to all the curves of the system, and would be called their *envelope.* On page 22 we find the parabola as the envelope of a series in which the right line appears as the variable curve. Similarly we find the parallel curve to the lemniscate (page 66) as the envelope of all circles of the same radius, whose centres are on that curve.

Analogously, a surface may be defined as the envelope of the various positions taken by another surface. Thus the right cone is the envelope of all the possible planes containing a given point on a line and making a constant angle with the line.

336. *Lines and surfaces regarded as the result of continuous motion.* By the conception with which the draughtsman is mainly concerned, a line is considered as generated by the motion of a *point;* a *surface* by the motion of a *line.* He regards a straight line as generated by a point moving always in the same direction;* a curve as the path of a point whose direction of motion changes continually; a plane as the surface generated by a moving straight line which glides along a fixed straight line while passing always through a fixed point.

337. A *curve* is *plane* if any four consecutive positions of its generating point lie in one plane.

338. A *space curve* or *curve of double curvature* or *non-plane curve* is any curve not plane; that is, a curve generated by a moving point, no four consecutive positions of which lie in the same plane.

Chief among space curves is the *ordinary helix,* whose construction is given in Art. 120. After it may be mentioned the *conical helix* (see Art. 191) and the *spherical epicycloid,* the latter being the theoretical outline of the teeth of bevel gears, and generated by the motion of a point on an element of a cone which rolls on another cone having the same slant height, their vertices also being common.

The spherical epicycloid evidently lies on the surface of a sphere whose radius equals an element of either cone.

339. Whether it be straight or curved, the moving *line* that generates a *surface* is called the *generatrix,* and in *any* of its positions is an *element* of the surface. Its motion may be either of *revolution* or *transposition.*

340. *Revolution* implies a straight line called an *axis,* which may or may not be in the same plane with the revolving line. Each point of the moving line describes a circle whose plane is perpendicular to the axis and whose centre is on the axis.

Any plane *containing the axis* of a surface of revolution is called a *meridian plane,* and cuts the surface in a *meridian curve.*

All planes *perpendicular to the axis* of a surface of revolution intersect it in circles called *parallels,* the smallest of which, so long as of finite radius, is called the *circle of the gorge.*

Any surface of revolution may also be generated as a surface of transposition.

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*Frequently also defined as *the shortest distance between two points.* If we take into account the generalized notions of modern geometry, as to *direction* and *distance,* we might here dispense with those terms and define a *straight line* as *the line which is completely determined by two points;* a *curve* as *any line not straight;* a *plane* as *the surface containing all straight lines connecting its points by pairs; parallel straight lines* as *non-intersecting straight lines in a plane*

341. *Transposition* may be defined as any motion other than revolution, and involves certain guiding *lines* called *directrices*, along which the generatrix glides; and, frequently, certain *surfaces* called *directors*, with respect to whose elements the generatrix of the new surface takes definite positions.

The surface-director most frequently occurring is the *cone*, including its special case, the *plane*. Every element of a cone-director will have a parallel element on the other surface.

### RULED SURFACES.

342. Whether the motion be of revolution or transposition, any surface that can have a *right line generatrix* is called a *ruled surface*. In contradistinction all others are called *double curved surfaces*, since *any* plane section thereof must be a curve.

As to *curvature*, ruled surfaces other than the *plane* are called *single curved*.*

343. The motion of a right line involves the consideration of "conditions of restraint" and "degrees of freedom." If it is to move so as always to intersect another line, the generatrix is under *one* condition of restraint, but, as to motion, has still two degrees of freedom. Require it in all its positions to intersect *two* lines and it has still one degree of freedom. Impose a third condition, either to move so as always to intersect each of three lines—either straight or curved—or, while intersecting two lines, to be parallel to some element of a surface-director, or to be tangent to a given surface, and we have then determined the line in position *thus far*, that we have located it upon a certain surface. Impose one more—a fourth condition—and the limit of *motion* is reached, the line becoming a particular line on that particular surface.

344. *Developable surfaces.* A surface which can be directly rolled out or unfolded upon a plane, without undergoing distortion, is called a *developable surface*.

345. *Plane-sided figures* evidently belong to this division, and after the *pyramid* and *prism* the more important are the *regular polyhedrons*, or solids bounded by *equal, regular polygons* whose planes are all inclined to each other at the same angle. Of these, *five* are convex to outer space, i.e., no face produced will cut into the solid. They are the *tetrahedron, octahedron* and *icosahedron*, bounded, respectively, by *four, eight* and *twenty* equal, equilateral *triangles;* the *cube* or *hexahedron*, whose *six* faces are equal *squares;* and the *dodecahedron*, whose surface consists of *twelve* equal, regular, *pentagons*.

By allowing re-entrant angles the number of regular solids has been increased by *four*, named after their investigator, Poinsot. They are frequently also called *star polyhedra*, from their form.†

346. A *single curved surface* is called a *developable* or *torse* if *consecutive positions* of its rectilinear generatrix lie *in the same plane*.

If *not only consecutive* elements but *any pair* lie in the same plane we have either a *cone* or a *cylinder*.

When *only consecutive* elements lie *in the same plane* we have developable surfaces of great variety, but whose common characteristic is, that they are generated by a straight line moving so that in every position it is a *tangent* to a curve of double curvature.

Fig. 208.

ELLIPTICAL CONE.

That such a surface is developable may be thus demonstrated without reference to any figure. Suppose $a$, $b$, $c$ and $d$ to be four *consecutive* points of a curve of double curvature. The line joining $a$ and $b$ is—by the ordinary definition—a tangent to the curve. Similarly $bc$ and $cd$ are tangent and consecutive. Tangent $ab$ meets tangent $bc$ at $b$, and their plane is tangent at $b$ to the generated surface, since it contains

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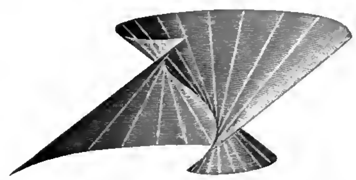
*Some writers make *single curved* synonymous with *developable*. (Art. 344).

†For their construction see Rouché et De Comberousse, *Traité de Géométrie*, Part II, Book VII.

two lines, each of which is tangent to the surface at the same point. Also $b\,c$ intersects $c\,d$ at $c$. But, since four consecutive points of a curve of double curvature cannot lie in the same plane, the tangent $a\,b$ does not meet tangent $c\,d$. The plane $a\,b\,c$ may, however, be rotated on $b\,c$ into coincidence with plane $b\,c\,d$, and this process repeated indefinitely, bringing the entire surface into one plane without disturbing the relative position of consecutive elements.

In Fig. 209 we have the *developable helicoid*, a representative surface of the family just described. Its elements (the white lines) are all tangent to a helix. In Arts. 186 and 187 its upper and lower outlines are described. Being right sections they are involutes.

Fig. 209.



Like the cone, complete surfaces of this family consist of two *nappes* or sheets. In the cone both nappes meet at the vertex. On other developables they meet on the curve whose tangents constitute the surface.

Could the nappes of such a surface be as readily separated as those of a cone, either of them could be as directly developed upon a plane by rolling contact.

The moving line generates both nappes simultaneously, its point of tangency separating it into the portions on the upper and lower nappes respectively.

Whatever the mathematical name for the curve of double curvature employed it has the following names as a line of the developable surface: (a) *cuspidal edge*, since any plane section of the surface will have a *cusp* or point, on that curve; (b) *edge of regression*, since along it the surface is most contracted.

347. The family of surfaces just described, as, in fact, all developable surfaces *but two*, *must* be surfaces of *transposition;* for a straight line can have *but three* positions with respect to an axis about which it is to *revolve:*

(a) it may be *parallel* to the axis, when a *cylinder of revolution* will result, *developable*, since consecutive elements lie in the same plane, because parallel;

(b) it may *intersect* the axis, giving the *cone of revolution*, *developable*, because consecutive elements lie in the same plane by virtue of intersecting;

(c) it may make an angle with the axis without intersecting it, giving consecutive positions not lying in the same plane.

348. *Warped surfaces* or *Scrolls.* A *warped surface* or *scroll* is a ruled surface in which *consecutive elements do not lie in the same plane.*

All warped surfaces *but one* must be surfaces of transposition: for we have seen, from the last article, that any straight line that generates a surface of *revolution* either *is* or *is not* in the same plane with the axis; and that if the former is the case a developable surface is generated: if, then, a *warped surface of revolution* be possible, it must be due to the revolution of a right line about an axis not in its plane—evidently the only other possible alternative. The proof that such motion fulfills the essential condition of a warped surface is as follows:

(a) Consecutive positions of the revolving line *do not intersect*, as—having *no* point *on* the axis—*each* point of it must describe a circle about the axis, and cannot, therefore, coincide with its preceding position;

(b) Consecutive positions *are not parallel*, for if parallel in space they would be in projection; while reference to Fig. 77, which is a plan and elevation of the surface in question, shows that they are projected as *consecutive tangents* to the projection of the circle of the gorge ($A\,B\,C\,D$), rendering parallelism impossible;

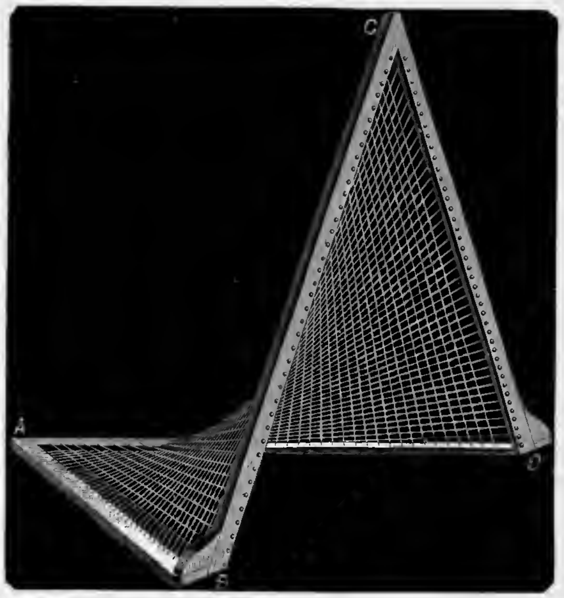
Neither intersecting nor being parallel, consecutive elements do not, therefore, lie in the same plane.

The warped surface of revolution is also called the *hyperboloid of revolution* since it may be generated by the revolution of its meridian curve—an hyperbola—about its conjugate (or imaginary) axis. If the asymptote to the hyperbola be simultaneously revolved with it, a surface called an asymptotic cone will be generated. The curves cut by a plane from the hyperboloid and its asymptotic cone will be of the same mathematical nature.

349. *Warped Surfaces of Transposition.* The more important of these to the graphicist are the *hyperbolic paraboloid;* the *elliptical hyperboloid* or *hyperboloid of one sheet; conoidal surfaces,* such as the *cono-cuneus of Wallis* and the *conoid of Plücker;* the *warped helicoid;* the *conchoidal hyperboloid of Catalan;* the *cylindroid of Frézier,* and the *warped arch,* also called the *corne de vache.*

Referring to Art. 343, regarding the number of conditions that may be imposed on a moving line, let us first consider all the possible surfaces having *three straight directrices.* Evidently such

Fig. 210.




HYPERBOLIC PARABOLOID.

Fig. 211.

HYPERBOLIC PARABOLOID.

Fig. 212.

ELLIPTICAL HYPERBOLOID.

directrices, no two of which lie in one plane, either *are* or *are not* parallel to some plane; for *any two* of them will invariably be parallel to some plane, and the third either *is* or *is not* parallel to that plane. Should the former be the case the resulting surface would be the *hyperbolic paraboloid,* which may also be defined as the warped surface having *two straight directrices and a plane director;* if, however, the directrices have the second position supposed, the surface generated is called the *hyperboloid of one sheet* or the *elliptical hyperboloid:* these two (and their special forms) evidently exhaust the possibilities as to surfaces with *only straight directrices.*

350. The hyperboloid and hyperbolic paraboloid are further interesting as being the only surfaces which may be *doubly-ruled,* that is, may have two different sets of rectilinear elements; for in such a surface it must be possible for *either directrix* of one set of elements to become the generatrix of a second set; hence, to become in its turn an *element,* each generatrix must be *straight,* which restricts the possibilities—as to number of surfaces—to the two named.

351. In a doubly-ruled warped surface *each* element of one generation, while intersecting no other elements of its own set, meets *all* of the other set; and *any three* of either set might be taken as directrices for the other.

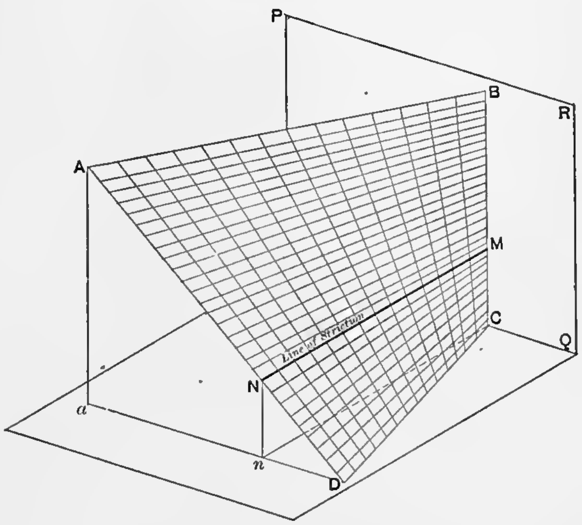
352. Each set of elements of the hyperbolic paraboloid is—by the second definition of the surface—parallel to its own plane director. Section-planes parallel to the line of intersection of the plane directors will cut the surface in *parabolas;* while other plane sections are *hyperbolas.*

The curvature of the hyperbolic paraboloid is called *anticlastic.* Its typical, saddle-like form is best illustrated by Fig. 211.

353. A *line of striction* is the line—straight or curved—containing the shortest distance between consecutive elements of a warped surface. For the hyperbolic paraboloid it is that element of one set which is perpendicular to the plane director of the other set of elements. (See $MN$ or $BC$, Fig. 213.)

354. A *conoidal surface* is generated by a line moving parallel to a plane director and so as always to intersect a fixed right line or axis, fulfilling, at the same time, an additional condition, such as the meeting of some fixed curve, or being tangent to a given fixed surface.

Fig. 213.




HYPERBOLIC PARABOLOID

WITH

LINES OF STRICTION, B C AND M N.

Fig. 214.

RIGHT CONOID.

THE CONO-CUNEUS

OF WALLIS.

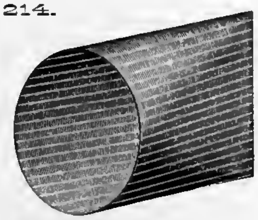
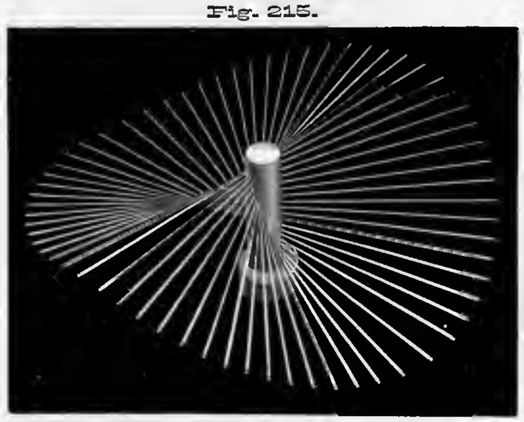


Fig. 215.



CONOID OF PLÜCKER.

When the axis or straight directrix is perpendicular to the plane director we have a *right conoid,* and the axis becomes the *line of striction.*

355. With a *circle* as the curved directrix; an axis *equal to, parallel to* and *directly opposite* a diameter of the circle; and with a plane director perpendicular to the axis, we would obtain the right conoid known as the *cono-cuneus of Wallis.* This surface (Fig. 214) has been employed with pleasing effect in architectural constructions, in towers, arches, and for the faces of wing-walls.

Plane sections parallel to the curved directrix are ellipses whose longer axes are equal to the straight directrix.

356. Another interesting surface of the same family is the *conoid of Plücker,** whose mechanical and kinematic properties are treated by Prof. R. S. Ball in his *Theory of Screws;* also by Clifford in his *Elements of Dynamic.* The name *cylindroid,* under which it appears in those works, was suggested by Cayley, but had much earlier been pre-empted by Frézier for the surface defined in Art. 360.

*See Plücker's *Neue Geometrie des Raumes,* p. 97; also Mannheim's *Géométrie Descriptive,* p. 435.

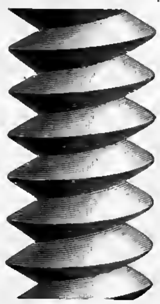
Fig. 215 represents a model of the conoid under consideration; but for the exact mathematical surface "the diameter of the central cylinder must be conceived to be evanescent, and the radiating wires must be extended to infinity." (*Ball*).

Were $AB$ and $CD$ (Fig. 213) to be the common directrices of a series of hyperbolic paraboloids whose plane directors were various positions of $PRQ$, rotated about their common line of striction $BC$, the other lines of striction would be elements of a conoid of Plücker.

The same surface will result if a right line be moved so that, while perpendicular to and intersecting the axis of a circular cylinder, it should follow a double-curved directrix obtained by wrapping around the cylinder a *sinusoid* (Art. 171) or *harmonic curve*, two waves of which reach once around.

357. The *warped helicoid* has for its directrices an ordinary helix and its axis, the generatrix making a constant angle with the latter, although not necessarily intersecting it. A cone director might obviously be the third condition imposed on the generatrix.

Fig. 216.

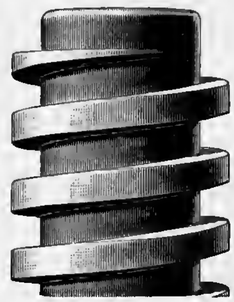


OBLIQUE HELICOIDS.

Fig. 217.

RIGHT HELICOID.

Fig. 218.



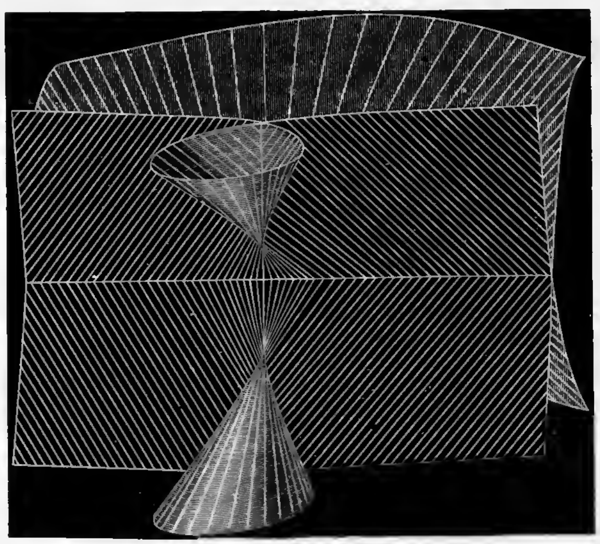
RIGHT HELICOIDS.

If the elements intersect the axis, and at an acute angle, the surface is called the ordinary *oblique helicoid*. It is the acting surface of triangular-threaded screws, and of many screw propellers.

Plane sections perpendicular to the axis of this surface are Archimedean spirals. (Art. 188).

358. When the elements intersect the axis at right angles the cone director becomes a plane

Fig. 219.



CONCHOIDAL HYPERBOLOID OF CATALAN.

director, and the surface is called the *right* helicoid, familiar to all as the under surface of a spiral staircase. It is the acting surface of a square-threaded screw and, usually, of screw propellers.

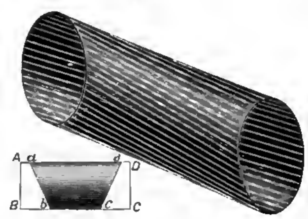
The right helicoid may evidently be classed among conoidal surfaces, since it has a plane director. Its axis is—like that of the cono-cuneus—a line of striction.

359. *The conchoidal hyperboloid of Catalan* (Fig. 219), has two non-intersecting, rectilinear directrices—one horizontal, the other vertical—the generatrix making a constant angle with the latter.

Planes parallel to both directrices will cut hyperbolas from the surface, while horizontal sections will be conchoids. (See Arts. 193 and 196).

360. *The cylindroid of Frézier* has a plane director and two curved directrices. In its usual form it may be imagined to be thus derived from a cylinder of revolution: Suppose a cylinder *A B C D* (Fig. 220) on H and parallel to V—the plane director; for curved directrices employ the ellipses cut from the cylinder by non-parallel, vertical, section-planes, *a b*, *c d*, taken on opposite sides of some vertical right section; if *one* of these ellipses be shifted *vertically, in its own plane*, the lines joining the new positions of its points with their former points of connection on the other directrix will be elements of a cylindroid.

Fig. 220.




CYLINDROID OF FRÉZIER.

This surface has been suggested for the soffit of a descending arch.

Any plane *containing* the line in which the planes of the curved directrices intersect will cut *congruent*[1] *curves* from the cylinder and cylindroid; while planes *parallel to* such line will cut plane sections of the same *area*.

361. *The warped arch* or *corne de vache* has three linear directrices, one straight, the others curved; the latter are equal circles in parallel planes, while the straight directrix is perpendicular to the planes of the circles and passes through the middle point of the line joining their centres.

Fig. 221.

CORNE DE VACHE.

In oblique or skew arch construction one of the best known methods[2] is that in which the soffit of the arch is a *corne de vache*, for which, obviously, only one-half of the surface would be employed. (Fig. 221).

DOUBLE CURVED SURFACES.

362. *Double Curved Surfaces* are surfaces that cannot be generated by a right line.

363. *Double Curved Surfaces of Revolution.* The *sphere* is the most familiar example under this head, the generatrix being a semi-circle and the axis its diameter. After it come the *ellipsoids*—the *prolate spheroid* and the *oblate spheroid*—generated by rotating an ellipse about its major or minor axis respectively; the *paraboloid of revolution*, generatrix—a parabola, axis—that of the curve; the *hyperboloid of revolution of separate nappes*, formed by rotating the two branches of an hyperbola about their transverse axis; and the *torus*—annular or not—generated by revolving a circle about an axis in its plane but not a diameter. (See Fig. 222; and also Arts. 112–114).

Fig. 222.

THE TORUS.

364. The revolution of other plane curves—as the involute, tractrix, conchoid, gives double curved surfaces of frequent use in architectural constructions and the arts.[3]

365. *Double Curved Surfaces of Transposition.* Of the innumerable surfaces possible under this head we need only mention here the *serpentine*, generated by a sphere whose centre travels along a helix; the *ellipsoid of three unequal axes*, which would result from turning an ellipse about one of its axes in such manner that, while remaining an ellipse, its other axis should so vary in length that its

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[1] Congruent figures, if superposed, will coincide throughout.

[2] For a comparison of the relative merits of these methods refer to *Skew Arches*, by E. W. Hyde (Van Nostrand's Science Series, No. 15). For full treatment of the plane sections see Wiener's *Darstellende Geometrie.*

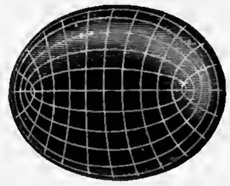
[3] See Note, p. 64; also Art. 203.

extremities would trace a second ellipse; the *elliptical paraboloid*, whose plane sections perpendicular to the axis are ellipses, while sections containing the axis are parabolas; the *elliptical hyperboloid of one nappe*, generated by turning a variable hyperbola about its real axis so that its arc shall follow an elliptical directrix; the *elliptical hyperboloid of two nappes*, analogous to the two-napped hyperboloid

Fig. 223.

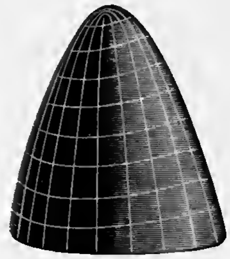
SERPENTINE.

Fig. 224.



ELLIPSOID.

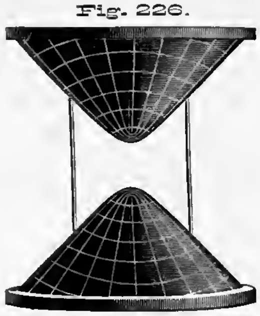
Fig. 225.



ELLIPTICAL HYPERBOLOID

of revolution, but having elliptical instead of circular sections perpendicular to its axis; and the *cyclide*,* whose lines of curvature (see Art. 381) are all circles, and each of whose normals intersects two conics—an ellipse and hyperbola—whose planes are mutually perpendicular, and having the foci of each at the extremities of the transverse axis of the other.

Fig. 226.



ELLIPTICAL HYPERBOLOID OF TWO NAPPES.

Fig. 227.

THE CYCLIDE OF DUPIN.

The cyclide is the envelope of all spheres (a) having their centres on one of the conics and (b) tangent to any sphere whose centre is on the other. The torus is a special form of cyclide.

TUBULAR SURFACES.—QUADRIC SURFACES OR CONICOIDS.

366. Among the surfaces we have described, the *torus* and *serpentine* belong to the family called *tubular*, since each is *the envelope of a sphere of constant radius.*

367. Surfaces whose plane sections are invariably conics are called *conicoids* or *quadrics.* These are the cone, cylinder and sphere; ellipsoids; hyperboloids of one and two nappes; elliptic and hyperbolic paraboloids. In theory, *all* conicoids are *ruled* surfaces; but on some the right lines are imaginary, while they are real on the cone, cylinder, hyperbolic paraboloid and warped hyperboloid.

TANGENTS AND NORMALS TO CURVES.—TANGENT CURVES.

368. A *tangent* to a curve is a right line joining two *consecutive* points of the curve. If tangent at infinity it is called an *asymptote.*

A *normal* to a curve is a right line perpendicular to the tangent at the point of tangency.

Both tangent and normal to a *plane curve* lie in its plane.

*Two curves are tangent* to each other when they have the same tangent line at any common point.

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*For mathematical, optical and other properties of the cyclide see Salmon's *Geometry of Three Dimensions*, and the writings of J. Clerk Maxwell.

369. If a curve is the intersection of a surface by a plane, the tangent to it at any point will be the intersection of the plane of the curve by the tangent plane to the surface at the given point.

The tangent at any point of a *non-plane* curve, when the latter is the intersection of two surfaces, is the line of intersection of two planes, each of which is tangent—at the given point—to one of the surfaces.

### LINES AND PLANES, TANGENT AND NORMAL TO SURFACES.

370. A straight line joining two consecutive points of a surface is a *tangent* to it.

371. *A tangent plane* to a surface at any point is the locus of all the right lines tangent to the surface at that point.

372. A right line perpendicular to a tangent plane at the point of tangency is a *normal* to the surface.

373. Any plane containing the normal cuts from the surface a *normal section.*

### TANGENT PLANES TO RULED SURFACES.

374. Since any element of a ruled surface fulfills the condition (Art. 370) necessary to make it a tangent to the surface, it may be taken as one of the two right lines necessary to determine a plane, tangent to the surface at any point of the element.

For a *doubly ruled* surface the two elements through the point would determine the tangent plane.

375. In general, the line which—with an element—would determine a tangent plane to a ruled surface at a given point, would be a tangent, at that point, to *any curve* passing through the latter and lying on that surface.

376. *A plane, tangent to a developable surface* at any point, would, therefore, be determined by the element containing the point, and, preferably, by a tangent to the base at the extremity of the element, since to such a surface a tangent plane has *line* and not merely *point* contact.

The element of tangency belongs to both the nappes which constitute a complete cone, developable helicoid or analogous surface; but the plane that is tangent to the surface along so much of the element as lies on one nappe is a secant plane to the other nappe.

377. *The tangent plane at any point of a warped surface* is found by Arts. 374 and 375, and is also, usually, a secant plane, tangency being along an entire element only in special cases. (See Art. 469).

### TANGENT PLANES TO DOUBLE CURVED SURFACES.

378. *A tangent plane to a double curved surface* has, usually, but one point of contact with it.

If a normal to the surface can readily be drawn then the tangent plane may be determined most simply on the principle that it will be perpendicular to the normal, at the given point. This method is especially applicable in problems of tangency to a *sphere*, since the radius to the point of tangency is the normal to the surface; also to any *tubular surface*, since such may be regarded as generated by the motion of a sphere, and at any point of the circle of contact of sphere and tubular surface they would have a common tangent plane.

*In general*, by taking the *two* simplest curves that could be drawn through the point of desired tangency and upon the surface, the tangent plane would be determined by the tangents to these curves at that point. Methods are given in Chapter V for drawing tangents to the more important mathematical curves, among which we would find nearly all of the plane sections of the surfaces defined in this chapter.

TANGENT SURFACES. — INTERSECTING SURFACES.

379. (a) *Two surfaces are tangent* to each other at a given point if at that point they have a common tangent plane. Any secant plane passing through a point of tangency of two tangent surfaces will cut them in sections that have a common tangent line.

(b) *Developable surfaces will be tangent* along a common element if they have a common tangent plane at one point of such element, since the element is one of the determining lines of the tangent plane to each surface.

(c) *Raccordment*, or the mutual tangency of *warped surfaces* along a common element, exists when at three different points of their common element they have common tangent planes.

(d) *Double curved surfaces* usually have *point* contact only, with single or double curved surfaces to which they are tangent. When, however, the tangent surfaces are con-axial and also surfaces of revolution they will have a circle of tangency; and with surfaces of transposition a curve of tangency is also possible. In any case, however, condition (a) above must be fulfilled.

(e) *The line of interpenetration of two intersecting surfaces* may be found by cutting them by a series of auxiliary surfaces; the two sections of the former, cut by any one of the latter, will meet (if at all) in points of the desired line of intersection. The surfaces should be so located with respect to H and V as to facilitate the drawing of the auxiliary sections; and the latter should, if possible, be straight lines or circles, in preference to other forms less easy to represent.

RADII AND LINES OF CURVATURE.— OSCULATORY CIRCLE AND PLANE.— GEODESICS.

380. *Osculatory circle.—Radius of curvature.—Osculating plane.* A circle is *osculatory* to a non-circular curve when it has three consecutive points in common with it. Its radius is called the *radius of curvature* of the curve, for the middle one of the three points. When a circle is osculatory to a *non-plane* curve its plane is called an *osculating plane.*

381. *Line of curvature.* It is ascertained by analysis that among all the possible normal sections at any point of a surface *two* may be found, mutually perpendicular, whose radii of curvature are respectively the maximum and minimum for that point; such sections are called *principal sections*, and their radii the *principal radii* for that point.

If upon any surface a line be so drawn that the tangent to it at any point lies in the direction of one of the principal sections at that point, the line is called a *line of curvature.*

Since at every point of a surface two principal sections are possible, there may also be drawn through each point *two* lines of curvature, intersecting each other at right angles. Such curves are shown in white lines on several of the preceding figures illustrating mathematical surfaces.

382. *Geodesics.* At every point of a *geodesic line* on a surface the osculating plane is normal to the surface.

Either the greatest or shortest distance between two points of a surface would be measured on the geodesic passing through them.

Since the maximum or minimum distance between two points on a *sphere* would be measured on the *great circle* containing them, such circle would be the geodesic on that surface.

The geodesic between two points on a cylinder would be a *helix.* On any other developable surface it would obviously be the space-form of the straight line which joined the given points on the *development* of the surface.

## *CHAPTER X.*

### PROJECTIONS AND INTERSECTIONS BY THE THIRD-ANGLE METHOD.—THE DEVELOPMENT OF SURFACES FOR SHEET METAL PATTERN MAKING.—PROJECTIONS, INTERSECTIONS AND TANGENCIES OF DEVELOPABLE, WARPED AND DOUBLE CURVED SURFACES, BY THE FIRST-ANGLE METHOD.

383. The mechanical drawings preliminary to the construction of machinery, blast furnaces, stone arches, buildings, and, in fact, all architectural and engineering projects, are made in accordance with the principles of Descriptive Geometry. When fully dimensioned they are called *working drawings.*

The object to be represented is supposed to be placed in either the first or the third of the four angles formed by the intersection of a horizontal plane, H, with a vertical plane, V. (Fig. 228).

The representations of the object upon the planes are, in mathematical language, *projections*,* and are obtained by drawing perpendiculars to the planes H and V from the various points of the object, the point of intersection of each such projecting line with a plane giving a *projection* of the original point. Such drawings are, obviously, not "views" in the ordinary sense, as they lack the perspective effect which is involved in having the point of sight at a finite distance; yet in ordinary parlance the terms *top view*, *horizontal projection* and *plan* are used synonymously; as are *front view* and *front elevation* with *vertical projection*, and *side elevation* with *profile view*, the latter on a plane perpendicular to both H and V and called the *profile plane.*

Until the last decade of the first century of Descriptive Geometry (1795–1895) problems were solved as far as possible in the first angle. As the location of the object in the third angle — that is, below the horizontal plane and behind the vertical — results in a grouping of the views which is in a measure self-interpreting, the *Third Angle Method* is, however, to a considerable degree supplanting the other for machine-shop work.

The advantageous grouping of the projections which constitutes the only—though a quite sufficient—justification for giving it special treatment, is this: The front view being always the *central* one of the group, the top view is found *at the top;* the view of the right side of the object appears *on the right;* of the left-hand side *on the left,* etc. Thus, in Fig. 228 (a), with the hollow block $BDFS$ as the object to be represented, we have $ades$ for its horizontal projection, $c'd'e'f'$ for its vertical projection, $f''e''s''x''$ for the side elevation; then on rotating the plane H clockwise on G.L. into coincidence with V, and the profile plane $P$ about $QR$ until the projection $f''e''s''x''$ reaches $f'''e'''s'''x'''$, we would have that location of the views which has just been described.

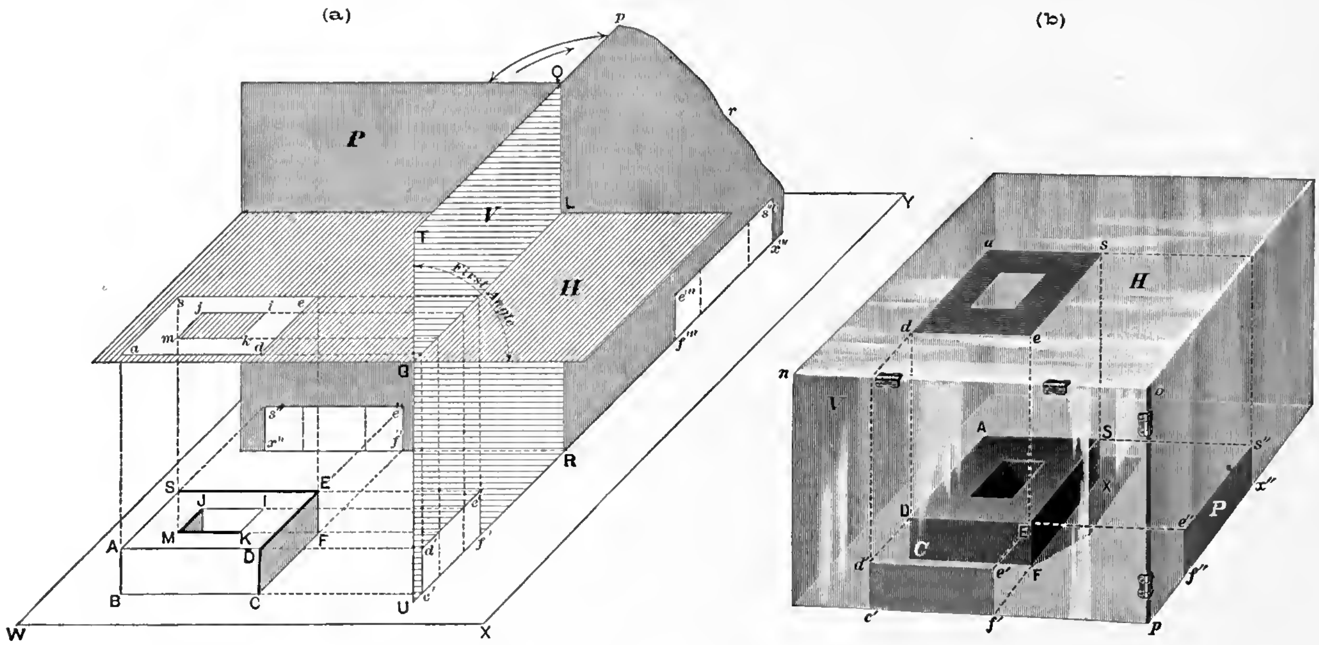
The lettering shows that each projection represents that side of the object which is toward the plane of projection.

384. The same grouping can be arrived at by a different conception, which will, to some, have advantages over the other. It is illustrated by Fig. 228 (b), in which the same object as before is

*For the convenience of those who have to take up this subject without previous study of Descriptive Geometry the Third-Angle section of this chapter is made complete in itself, by the re-statement of the principles involved and which have been treated at somewhat greater length in the previous chapter; although a review of such matter may be by no means disadvantageous to those who have already been over the fundamentals.

supposed to be surrounded by a system of mutually-perpendicular transparent planes, or, in other words, to be in a box having glass sides, and on each side a drawing made of what is seen through that side, excluding the idea, as before, of perspective view, and representing each point by a per-

Fig. 228.




pendicular from it to the plane. The whole system of box and planes, in the wood-cut, is rotated 90° from the position shown in Fig. 228 (a), bringing them into the usual position, in which the observer is looking perpendicularly toward the vertical plane.

Fig. 229.

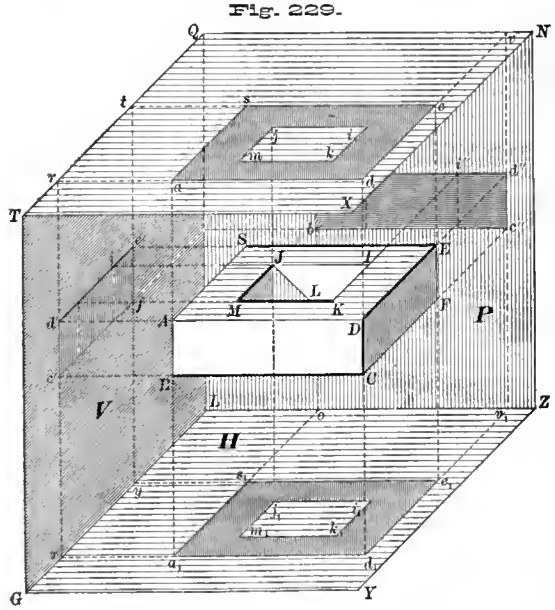
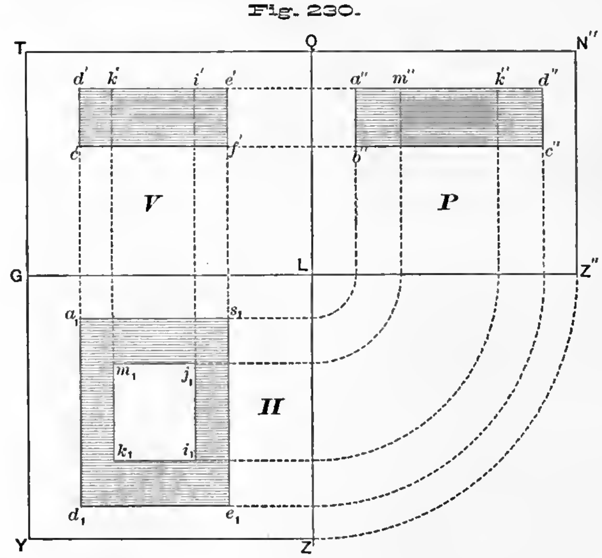



Fig. 230.



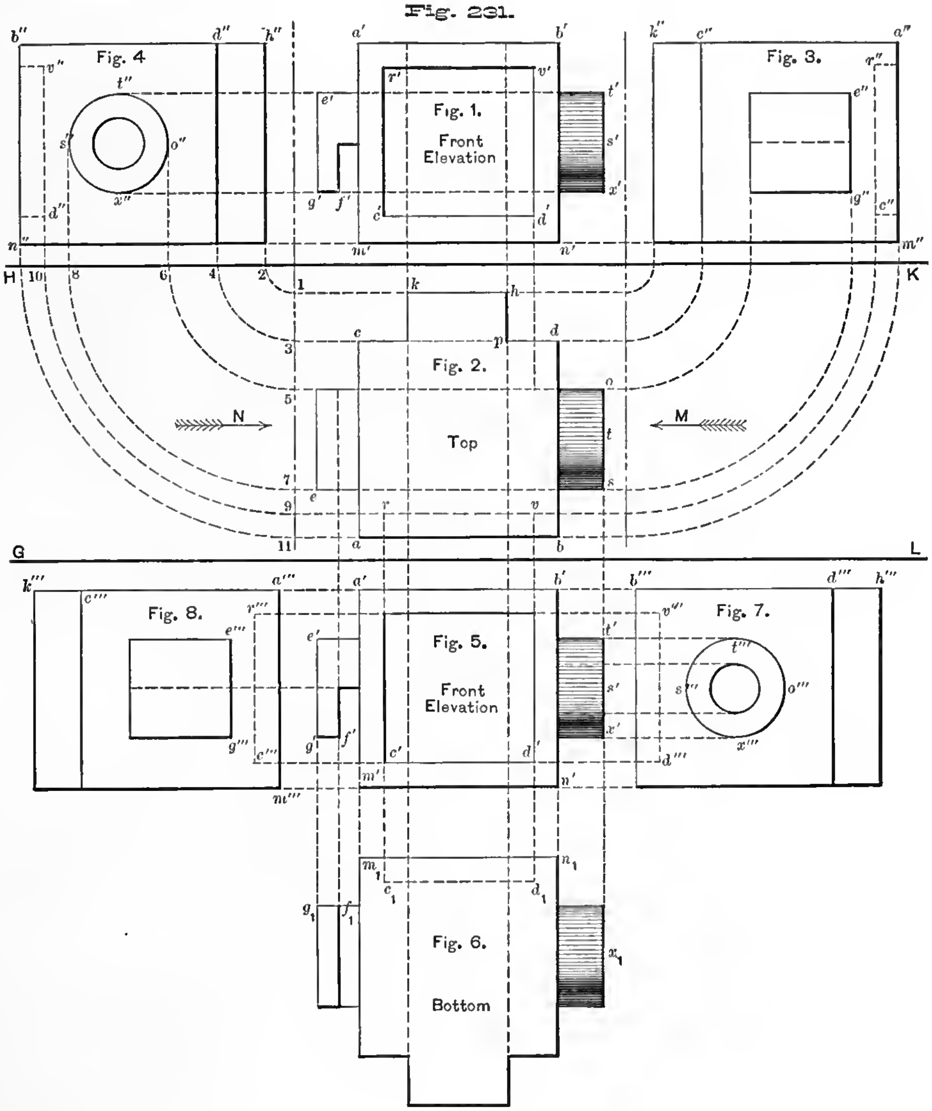

385. In Fig. 229 we may illustrate either the First or the Third Angle method, as to the top view of the object; $a\,d\,e\,s$ in the upper plane being the *plan* by the latter method, and $a_1\,d_1\,e_1\,s_1$ by the former.

Disregarding $QTXN$ we have the object and planes illustrating the first-angle method throughout, the lettering of each projection showing that it represents the side of the object *farthest* from the plane, making it the exact reverse of the third-angle system.

*In the ordinary representation* the same object would be represented simply by its three views as in Fig. 230. In the elevations the short-dash lines indicate the invisible edges of the hole.

The arcs show the rotation which carries the profile view into its proper place.

Fig. 231.




386. For the sake of more readily contrasting the two methods a group of views is shown in Fig. 231, all above G. L., illustrating an object by the First Angle system, while all below $HK$ represents the same object by the Third Angle method.

When looking at Figures 1, 2, 3 and 4 the observer queries: What is the object, *in space*, whose *front* is like Fig. 1, *top* is like Fig. 2, *left* side is like Fig. 3 and *right* side like Fig. 4?

For the view of the left side he might imagine himself as having been at first between $G$ and $H$, looking in the direction of arrow $N$, after which both himself and the object were turned, together

to the right, through a ninety-degree arc, when the same side would be presented to his view in Fig. 3. Similarly, looking in the direction of the arrow *M*, an equal rotation to the left, as indicated by the arcs 1-2, 3-4, 5-6, etc., would give in Fig. 4 the view obtained from direction *M*. His mental queries would then be answered about as follows: Evidently a cubical block with a rectangular recess—$r'v'd'c'$—in front; on the rear a prismatic projection, of thickness $ph$ and whose height equals that of the cube; a short cylindrical ring projecting from the right face of the cube; an angular projecting piece on the left face.

In Fig. 2 the line $rv$ is in short dashes, as in that view the back plane of the recess $r'v'd'c'$ would be invisible. In Fig. 4 the back plane of the same recess is given the letters, $v''d''$, of the edge nearest the observer from direction *M*.

*To illustrate the third angle method* by Fig. 231 we ignore all above the line *HK*. In Fig. 5 we have the same front elevation as before, but *above it* the view of the *top; below it* the view of the *bottom* exactly as it would appear were the object held before one as in Fig. 5, then given a ninety-degree turn, around $a'b'$, until the under side became the front elevation.

Fig. 7 may as readily be imagined to be obtained by a shifting of the object as by the rotation of a plane of projection; for by translating the object to the right, from its position in Fig. 5, then rotating it to the left 90° about $b'n'$, its right side would appear as shown.

387. For convenient reference a general resumé of terms, abbreviations and instructions is next presented, once for all, for use in both the Third Angle and First Angle methods.

(1) H, V, P . . . . . . . . . . . . . the *horizontal*, *vertical* and *profile* planes of projection respectively.
(2) H-projector . . . . . . . . . . . the projecting line which gives the *horizontal projection* of a point.
(3) V-projector. . . . . . . . . . . . the projecting line giving the projection of a point on V.
(4) Projector-plane. . . . . . . . . . the profile plane containing the projectors of a point.
(5) h. p. . . . . . . . . . . . . . . . the *horizontal projection* or *plan* of a point or figure.
(6) v. p. . . . . . . . . . . . . . . . the *vertical projection* or *elevation* of a point or figure.
(7) h. t. . . . . . . . . . . . . . . . *horizontal trace*, the intersection of a line or surface with H.
(8) v. t. . . . . . . . . . . . . . . . *vertical trace*, the intersection of a line or surface with V.
(9) H-traces, V-traces. . . . . . . . plural of horizontal and vertical traces respectively.
(10) G. L. . . . . . . . . . . . . . . *ground line*, the line of intersection of V and H.
(11) V-parallel . . . . . . . . . . . . a line parallel to V and lying in a given plane.
(12) A horizontal. . . . . . . . . . . any horizontal line lying in a given plane.
(13) Line of declivity. . . . . . . . . the steepest line, with respect to one plane, that can lie in another plane.
(14) Rabatment . . . . . . . . . . . . revolution into H or V about an axis in such plane.
(15) Counter-rabatment or revolution . restoration to original position.

388. *For Problems relating solely to the Point, Line and Plane.*

*Given lines* should be fine, continuous, black; *required lines* heavy, continuous, black or red; *construction lines* in fine, continuous red, or short-dash black; *traces of an auxiliary plane*, or *invisible traces* of *any* plane, in dash-and-three-dot lines. —— - - - —— - - - —— - - - ——

*For Problems relating to Solid Objects.*

(1) *Pencilling.* Exact; generally completed for the whole drawing before any inking is done; the work usually from centre lines, and from the larger—and nearer—parts of the object to the smaller or more remote.

(2) *Inking of the Object.* Curves to be drawn before their tangents; fine lines uniform and drawn before the shade lines; shade lines next and with one setting of the pen, to ensure uniformity. On *tapering shade lines* see Art. 111.

(3) *Shade Lines. In architectural work* these would be drawn in accordance with a given direction of light.

In *American machine-shop practice* the *right-hand* and *lower* edges of a *plane* surface are made shade lines if they separate it from *invisible* surfaces. Indicate *curvature* by *line-shading* if not otherwise sufficiently evident. (See Fig. 288).

(4) *Invisible lines of the object*, black, invariably, in dashes nearly one-tenth of an inch in length. - - - - - - - - - -

(5) *Inking of lines other than of the object.* When *no colors* are to be employed the following directions as to kind of line are those most frequently made. The lines may preferably be drawn in the order mentioned.

*Centre lines*, an alternation of dash and two dots.

*Dimension lines*, a dash and dot alternately, with opening left for the dimension.

*Extension lines*, for dimensions placed outside the views, in dash-and-dot as for a dimension line.

*Ground line*, (when it cannot be advantageously omitted) a continuous heavy line.

*Construction* and other explanatory lines in short dashes. - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

(6) *When using colors* the *centre, dimension and extension lines* may be fine, continuous, *red;* or the former may be *blue*, if preferred. *Construction lines* may also be red, in short dashes or in fine continuous lines.

Instead of using bottled inks the carmine and blue may preferably be taken directly from Winsor and Newton cakes, "moist colors." Ink ground from the cake is also preferable to bottled ink.

Drawings of developable and warped surfaces are much more effective if their elements are drawn in some color.

(7) *Dimensions and Arrow-Tips.* The *dimensions* should invariably be in *black*, printed *free-hand* with a writing-pen, and *should read in line with the dimension line they are on.* On the drawing as a whole the dimensions should read either from the bottom or right-hand side. *Fractions* should have a *horizontal* dividing line; although there is high sanction for the omission of the dividing line, particularly in a mixed number.

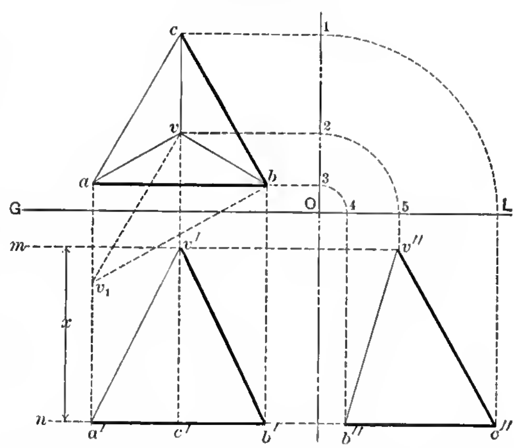
Extended Gothic, Roman, Italic Roman and Reinhardt's form of Condensed Italic Gothic are the best and most generally used types for dimensioning.

*The arrow-tips* are to be always drawn *free-hand*, in *black;* to touch the lines between which they give a distance; and to make an acute instead of a right angle at their point.

389. *Working drawing of a right pyramid; base, an equilateral triangle 0.9″ on a side; altitude, x.*

Draw first the equilateral triangle $a\,b\,c$ for the plan of the base, making its sides of the prescribed length. If we make the edge $a\,b$ perpendicular to the profile plane, $O\,1$, the face $v\,a\,b$ will then appear in profile view as the straight line $v''b''$. Being a *right* pyramid, with a *regular* base, we shall find $v$, the plan of the vertex, equally distant from $a$, $b$ and $c$; and $v\,a$, $v\,b$, $v\,c$ for the plans of the edges.

Fig. 232.

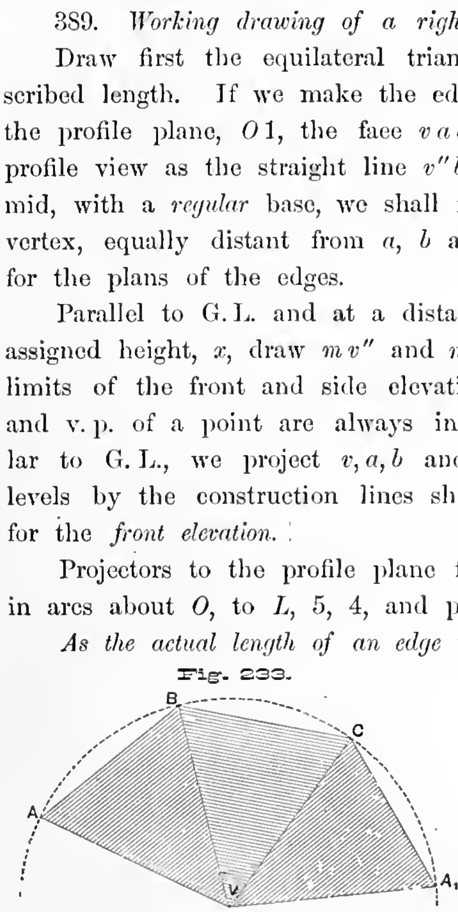


Parallel to G.L. and at a distance apart equal to the assigned height, $x$, draw $m\,v''$ and $n\,c''$ as upper and lower limits of the front and side elevations; then, as the h.p. and v.p. of a point are always in the same perpendicular to G.L., we project $v, a, b$ and $c$ to their respective levels by the construction lines shown, obtaining $v'.a'b'c'$ for the *front elevation.*

Projectors to the profile plane from the points of the plan give 1, 2, 3, which are then carried, in arcs about $O$, to $L$, 5, 4, and projected to their proper levels, giving the *side elevation*, $v''b''c''$.

*As the actual length of an edge* is not shown in either of the three views, we employ the following construction to ascertain it: Draw $v\,v_1$ perpendicular to $v\,b$, and make it equal to $x$; $v_1 b$ is then the real length of the edge, shown by rabatment about $v\,b$.

Fig. 233.



The *development* of the pyramid (Fig. 233) may be obtained by drawing an arc $A\,B\,C\,A_1$ of radius $= v_1 b$ (the true length of edge, from Fig. 232) and on it laying off the chords $A\,B$, $B\,C$, $C\,A_1$ equal to $a\,b$, $b\,c$, $c\,a$ of the plan; then $V\text{–}A\,B\,C\,A_1$ is the plane area which, folded on $V\,C$ and $V\,B$, would give a model of the pyramid represented.

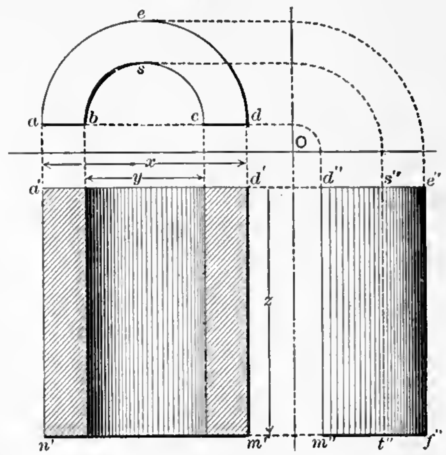
390. *Working drawing of a semi-cylindrical pipe:* outer diameter, $x$; inner diameter, $y$; height, $z$.

For the *plan* draw concentric semi-circles $a\,e\,d$ and $b\,s\,c$, of diameters $x$ and $y$ respectively, joining their extremities by straight lines $a\,b$, $c\,d$. At a distance apart of $z$ inches draw the upper and lower limits of the elevations, and project to these levels from the points of the plan.

In the side view the thickness of the shell of the cylinder is shown by the distance between $e''f''$ and $s''t''$—the latter so drawn as to indicate an invisible limit or line of the object.

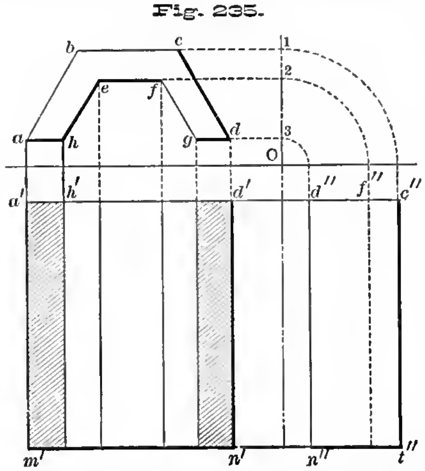
The line shading would usually be omitted, the shade lines generally sufficing to convey a clear idea of the form.

Fig. 234.




391. *Half of a hollow, hexagonal prism.* In a semi-circle of diameter $a\,d$ step off the radius three times as a chord, giving the vertices of the plan $a\,b\,c\,d$ of the outer surface. Parallel to $b\,c$, and at a distance from it equal to the assigned thickness of the prism, draw $e\,f$, terminating it on lines (not shown) drawn through $b$ and $c$ at 60° to $a\,d$. From $e$ and $f$ draw $e\,h$ and $f\,g$, parallel respectively to $a\,b$ and $c\,d$. Drawing $a'c''$ and $m't''$ as upper and lower limits, project to them as in preceding problems for the front and side elevations.

Fig. 235.

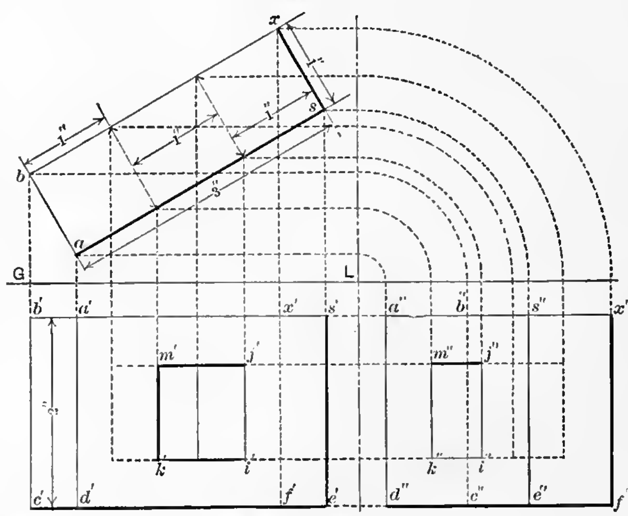



392. *Working drawing of a hollow, prismatic block, standing obliquely to the vertical and profile planes.*

Let the block be $2'' \times 3'' \times 1''$ outside, with a square opening $1'' \times 1'' \times 1''$ through it in the direction of its thickness. Assuming that it has been required that the two-inch edges should be vertical, we first draw, in Fig. 236, the plan $a\,s\,x\,b$, $3'' \times 1''$, on a scale of 1:2. The inch-wide opening through the centre is indicated by the short-dash lines.

For the elevations the upper and lower limits are drawn 2″ apart, and $a$, $b$, $s$, $x$, etc., projected to them. The elevations of the opening are between levels $m'm''$ and $k'k''$, one inch apart and equi-distant from the upper and lower outlines of the views. The dotted construction lines and the lettering will enable the student to recognize the three views of any point without difficulty.

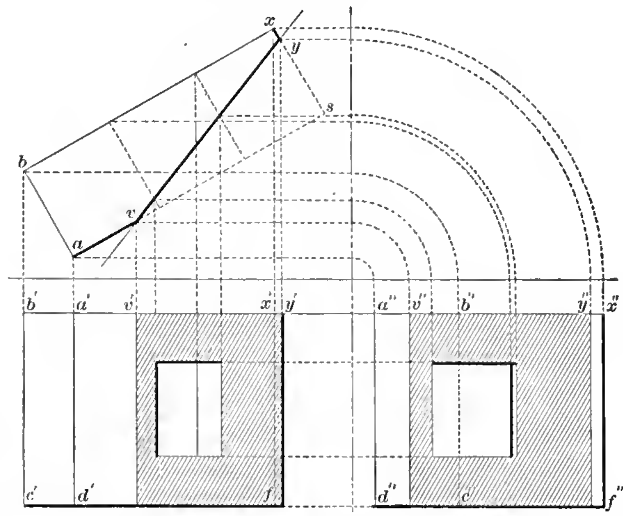
Fig. 236.




393. In Fig. 237 we have the same object as that illustrated by Fig. 236, but now represented as cut by a vertical plane whose horizontal trace is $v\,y$. The parts of the block that are actually cut by the plane are shown in section-lines in the elevations. This is done here and in some later examples merely to aid the beginner in understanding the views; but, *in engineering practice, section-lining is rarely done on views not perpendicular to the section plane.*

394. *Suppression of the ground line.* In machine drawing it is customary to omit the ground line, since the *forms* of the various views—which alone concern us—are independent of the distance of the object from an imaginary horizontal or vertical plane. We have only to remember that all elevations of a point are at the same level; and that if a ground line or trace of any vertical plane is wanted, it will be perpendicular to the line joining the plan of a point with its projection on such vertical plane. (Art. 286.)

Fig. 237.

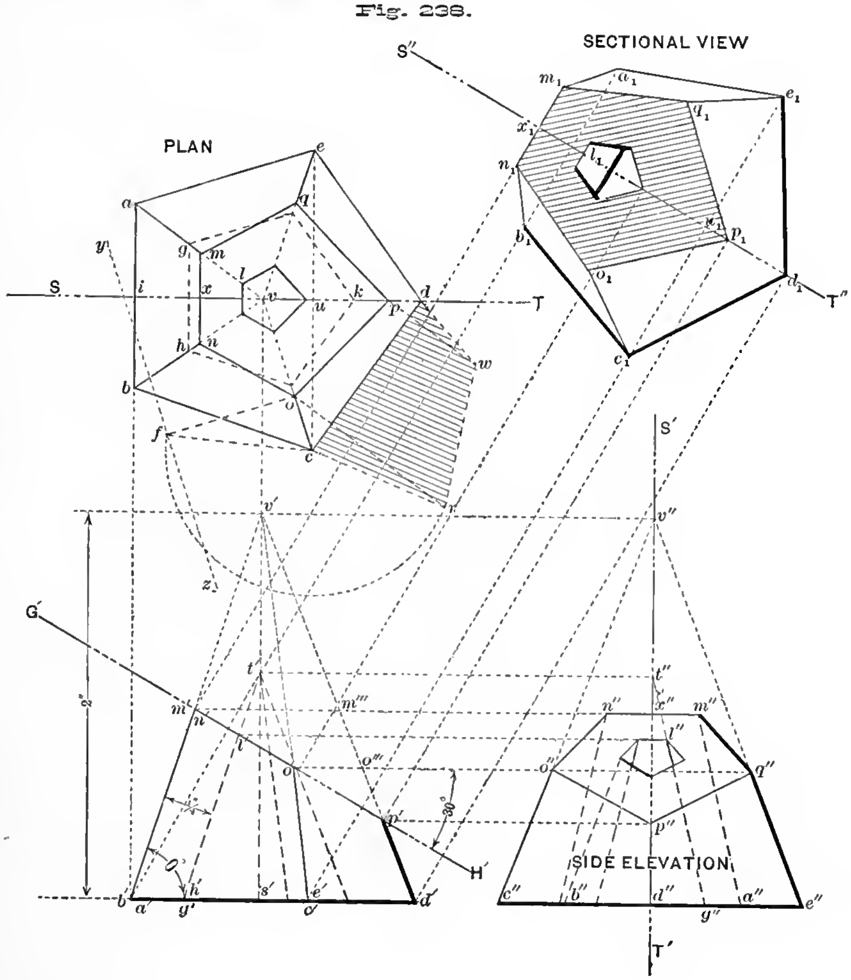


395. *Sections. Sectional views.* Although earlier defined (Art. 70), a re-statement of the distinction between these terms may well precede problems in which they will be so frequently employed.

When a plane cuts a solid, that portion of the latter which comes *in actual contact* with the cutting plane is called the *section*.

A *sectional view* is a view *perpendicular to the cutting plane*, and showing not only the section but also the object itself as if seen through the plane. When the cutting plane is *vertical* such a view is called a *sectional elevation;* when *horizontal*, a *sectional plan.*

Fig. 238.




396. *Working drawing of a regular, pentagonal pyramid, hollow, truncated by an oblique plane; also the development, or "pattern," of the outer surface below the cutting plane.* For data take the altitude at 2″; inclination of faces, $\theta°$ (meaning any arbitrary angle); inclination of section plane, 30°; distance between inner and outer faces of pyramid, $\frac{1}{4}$″.

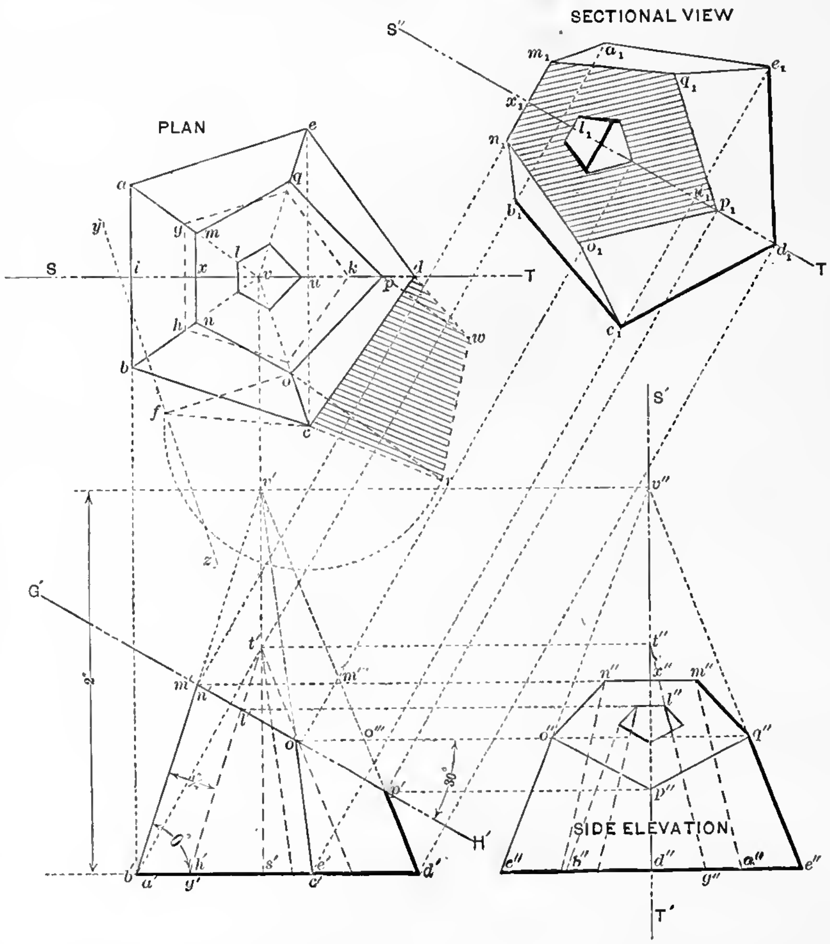
(1) Locate $v$ and $v'$ (Fig. 238) for the plan and elevation of the vertex, taking them sufficiently apart to avoid the overlapping of one view upon the other. Through $v$ draw the horizontal line $ST$, regarding it not only as a centre line for the plan but also as the h. t. of a central, vertical, reference plane, parallel to the ordinary vertical plane of projection.

(The student should note that for convenient reference Fig. 238 is repeated on this page.)

On the vertical line $v\,v'$ (at first indefinite in length) lay off $v's'$ equal to 2″, for the altitude (and axis) of the pyramid, and through $s'$ draw an indefinite horizontal line, which will contain the v. p. of the base, in both front and side views.

Draw $v'b'$ at $\theta°$ to the horizontal. It will represent the v. p. of an outer *face* of the pyramid, and $b'$ will be the v. p. of the edge $a\,b$ of the *base*. The base $a\,b\,c\,d\,e$ is then a regular pentagon circumscribed about a circle of centre $v$ and radius $v\,i = s'b'$. Since the angle $a\,v\,b$ is 72° (Art. 92) we get a starting *corner*, $a$ or $b$, by drawing $v\,a$ or $v\,b$ at 36° to $S\,T$, to intercept the vertical through $b'$. The plans of the edges of the pyramid are then $v\,a$, $v\,b$, $v\,c$, $v\,d$ and $v\,e$. Project $d$ to $d'$ and draw $v'd'$ for the elevation of $v\,d$; similarly for $v\,e$ and $v\,c$, which happen in this case to coincide in vertical projection.

Fig. 238.




For the inner surface of the pyramid, whose faces are at a perpendicular distance of ¼″ from the outer, begin by drawing $g'l'$ parallel to and ¼″ from the face projected in $b'v'$; this will cut the axis at a point $t'$ which will be the vertex of the inner surface, and $g't'$ will represent the elevation of the inner face that is parallel to the face $a\,v\,b — v'b'$; while $g\,h$, vertically above $g'$ and included between $v\,a$ and $v\,b$, will be the plan of the lower edge of this face. Complete the pentagon $g\,h$---$k$ for the plan of the inner base; project the corners to $b'd'$ and join with $t'$ to get the elevations of the interior edges.

*The section.* In our figure let $G'H'$ be the section plane, situated perpendicular to the vertical plane and inclined 30° to the horizontal. It intersects $v'd'$ in $p'$, which projects upon $v\,d$ at $p$. Similarly, since $G'H'$ cuts the edges $v'c'$ and $v'e'$ at points projected in $o'$, we project from the latter to $v\,c$ and $v\,e$, obtaining $o$ and $q$. A like construction gives $m$ and $n$. The polygon $m\,n\,o\,p\,q$ is then the plan of the outer boundary of the section.

The inner edge $g't'$ is cut by the section plane at $l'$, which projects to both $v\,h$ and $v\,g$, giving the parallel to $m\,n$ through $l$. The inner boundary of the section may then be completed either by determining all its vertices in the same way or on the principle that its sides will be parallel to those of the outer polygon, since any two planes are cut by a third in parallel lines.

The line $m'p'$ is the vertical projection of the entire section.

(2) *The side elevation.* This might be obtained exactly as in the five preceding figures, that is, by actually locating the side vertical, or *profile*, plane, projecting upon it and rotating through an arc of 90°. *In engineering practice,* however, *the method now to be described is in far more general use.* It does not do away with the profile plane, on the contrary presupposes its existence, but instead of actually locating it and drawing the arcs which so far have kept the relation of the views constantly before the eye, it reaches the same result in the following manner: A vertical line $S' T'$ is drawn at some convenient distance to the right of the front elevation; the distance, from $S T$, of any point of the plan, is then laid off horizontally from $S' T'$, at the same height as the front elevation of the point. For, as earlier stated, $S T$ was to be regarded as the horizontal trace of a vertical plane. Such plane would evidently cut a *profile* plane in a vertical line, which we may call $S' T'$, and let the $S' T'$ of our figure represent it after a ninety-degree rotation has occurred. The distances of all points of the object, to either the *front* or *rear* of the vertical plane on $S T$, would, obviously, be now seen as distances to the *left* or *right*, respectively, of the trace $S' T'$, and would be directly transferred with the dividers to the lines indicating their level. Thus, $e''$ is on the *level* of $e'$, but is to the *right* of $S' T'$ the same distance that $e$ is *above* (or, in reality, *behind*) the plane $S T$; that is, $e'' d''$ equals $e u$. Similarly $d'' b''$ equals $i b$; $n'' x''$ equals $n x$.

It is usual, where the object is at all symmetrical, to locate these reference planes *centrally*, so that their traces, used as indicated, may *bisect* as many lines as possible, to make one setting of the dividers do double work.

(3) *True size of the section. Sectional view.* If the section plane $G' H'$ were rotated directly about its trace on the central, vertical plane $S T$, until parallel to the paper, it would show the section $m' p'$—$m n o p q$ in its true size; but such a construction would cause a confusion of lines, the new figure overlapping the front elevation. If, however, we transfer the plane $G' H'$—keeping it parallel to its first position during the motion—to some new position $S''' T'''$, and then turn it 90° on that line, we get $m_1 n_1 o_1 p_1 q_1$, the desired view of the section. The distances of the vertices of the section from $S''' T'''$ are derived from reference to $S T$ exactly as were those in the side elevation; that is, $m_1 x_1 = m x = m'' x''$. We thus see that one central, vertical, reference plane, $S T$, is auxiliary to the construction of two important views; $S' T'$ represents its intersection with the profile or side vertical plane, while $S''' T'''$ is its (transferred) trace upon the section plane $G' H'$. For the remainder of the sectional view the points are obtained exactly as above described for the section; thus $e' e_1 e_1$ is perpendicular to $S''' T'''$; $e_1 u_1$ equals $e u$, and $c_1 u_1$ equals $c u$.

(4) *To determine the actual length of the various edges.* The only edge of the original, uncut pyramid, that would require no construction in order to show its true length, is the extreme right-hand one, which—being parallel to the vertical plane, as shown by its plan $v d$ being horizontal—is seen in elevation in its true size, $v' d'$. Since, however, all the edges of the pyramid are equal, we may find on $v' d'$ the true length of any *portion* of some other edge, as, for example $o' e'$, by taking that part of $v' d'$ which is intercepted between the same horizontals, viz.: $o''' d'$.

Were we compelled to find the true length of $o' e'$, $o e$, independently of any such convenient relation as that just indicated, we would apply one of the methods fully illustrated by Figs. 183, 184 and 187, or the following "shop" modification of one of them: Parallel to the plan $o c$ draw a line $y z$, their distance apart to be equal to the *difference of level* of $o'$ and $e'$, which difference may be obtained from either of the elevations; from the plan $o$ of the higher end of the line draw the common perpendicular $o f$, and join $f$ with $e$, obtaining the desired length $f c$.

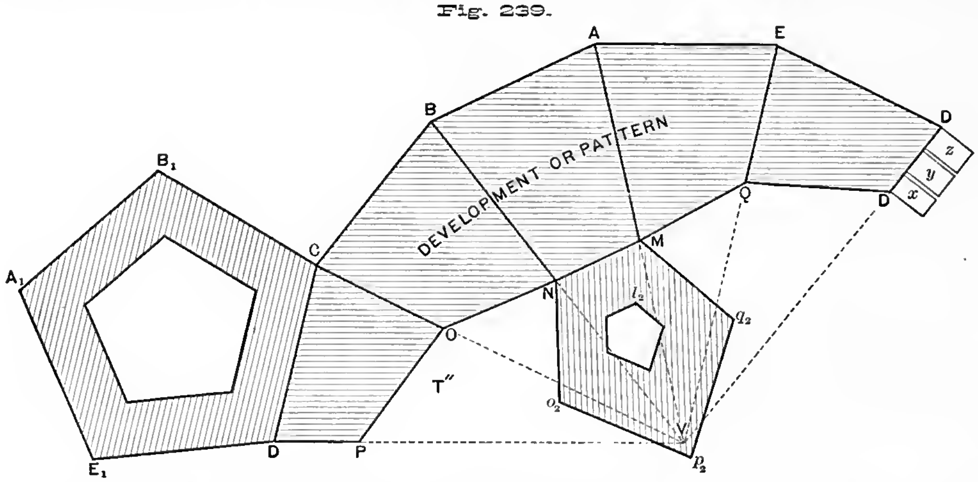
(5) *To show the exact form of any face of the pyramid.* Taking, for example, the face $o e d p$, revolve $o p$ about the horizontal edge $e d$ until it reaches the level of the latter. The actual distance

of $o$ from $c$, and of $p$ from $d$ will be the same after as before this revolution, while the paths of $o$ and $p$ during rotation will be projected in lines $o\,r$ and $p\,w$, each perpendicular to $c\,d$; therefore, with $c$ as a centre, cut the perpendicular $o\,r$ by an arc of radius $f\,c$—just ascertained to be the real length of $o\,c$, and, similarly, cut $p\,w$ by an arc of radius $d\,w = p'd'$; join $r$ with $c$, $w$ with $d$, draw $w\,r$ and we have in $c\,d\,w\,r$ the form desired.

(6) *The development of the outer surface of the truncated pyramid.* With any point $V$ as a centre (Fig. 239) and with radius equal to the actual length of an edge of the pyramid (that is, equal to $v'd'$, Fig. 238) draw an indefinite arc, on which lay off the chords $D\,C$, $C\,B$, $B\,A$, $A\,E$, $E\,D$, equal respectively to the like-lettered edges of the base $a\,b\,c\,d\,e$; join the extremities of these chords with $V$: then on $D\,V$ lay off $D\,P = d'p'$; make $C\,O = E\,Q = d'o''' =$ the real length of $c'o'$; also $B\,N = A\,M = d'm''' =$ the actual length of $a'm'$ and $b'n'$; join the points $P$, $O$, etc., thus obtaining the development of the outer boundary of the section. The pattern $A_1\,B_1\,C\,D\,E_1$ of the base is obtained from the *plan* in Fig. 238, while $N\,M\,q_2\,p_2\,o_2$ is a duplicate of the shaded part of the *sectional view* in the same figure.

(7) *In making a model* of the pyramid the student should use heavy Bristol board, and make allowance, wherever needed, of an extra width for overlap, slit as at $x$, $y$ and $z$ (Fig. 239). On this

Fig. 239.




overlap put the mucilage which is to hold the model in shape. The faces will fold better if the Bristol board is cut half way through on the folding edge.

397. For convenient reference the characteristic features of the Third Angle Method, all of which have now been fully illustrated, may thus be briefly summarized:

(a) The various views of the object are so grouped that the *plan* or *top view* comes *above* the front elevation; that of the bottom *below* it; and analogously for the projections of the right and left sides.

(b) Central, reference planes are taken through the various views, and, in each view, the distance of any point from the trace of the central plane of that view is obtained by direct transfer, with the dividers, of the distance between the same point and reference plane, as seen in some other view, usually the plan.

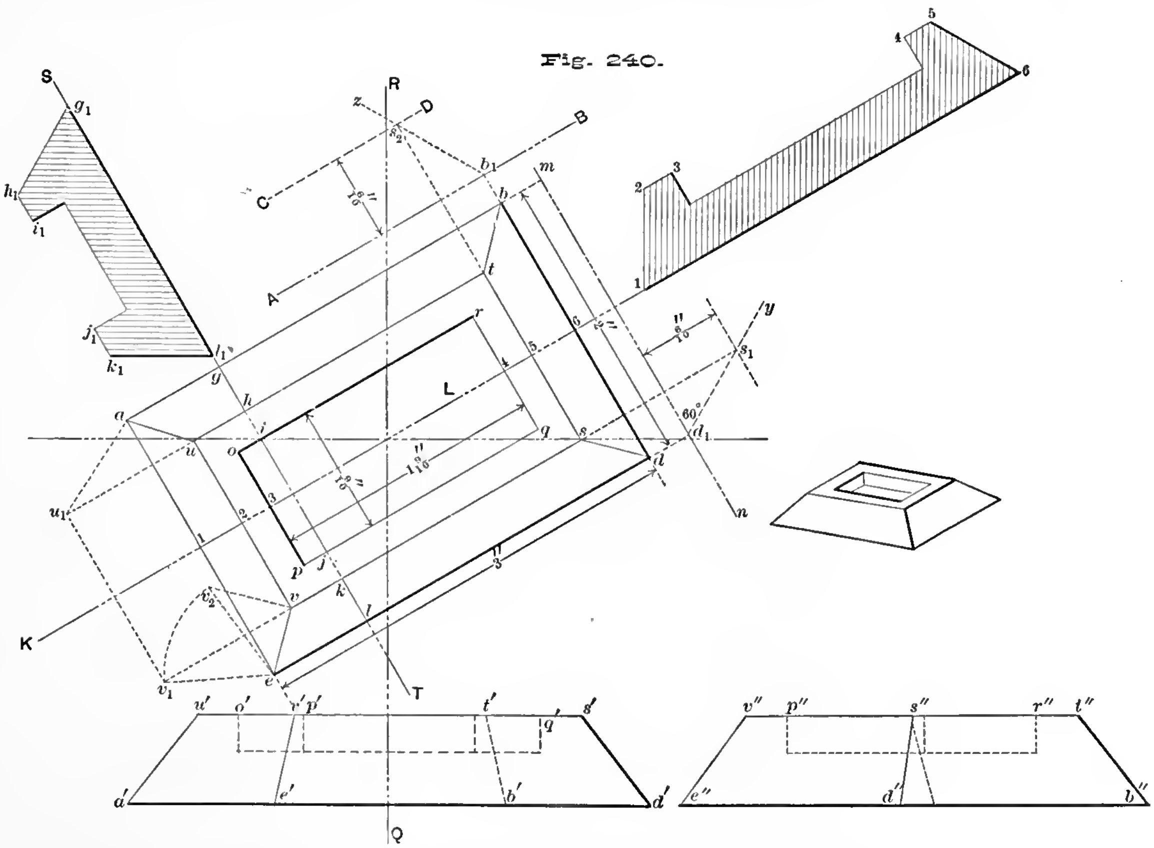
398. *To draw a truncated, pyramidal block, having a rectangular recess in its top;* angle of sides, 60°; lower base a rectangle $3'' \times 2''$, having its longer sides at 30° to the horizontal; total height $\frac{6}{10}''$; recess $1\frac{9}{10}'' \times \frac{9}{10}''$, and $\frac{1}{4}''$ deep. (Fig. 240.)

The small oblique projection on the right of the plan shows, pictorially, the figure to be drawn.

The plan of the lower base will be the rectangle $a\,b\,d\,e$, $3'' \times 2''$, whose longer edges are inclined $30°$ to the horizontal.

Take $A\,B$ and $m\,n$ as the H-traces of auxiliary, *vertical* planes, perpendicular to the side and end faces of the block. Then the sloping face whose lower edge is $d\,e$, and which is inclined $60°$ to H, will have $d_1\,y$ for its trace on plane $m\,n$. A parallel to $m\,n$ and $\frac{6}{10}''$ from it will give $s_1$, the auxiliary projection of the upper edge of the face $s\,v\,e\,d$, whence $s\,v$—at first indefinite in *length*—is derived, parallel to $d\,e$. Similarly the end face $b\,t\,s\,d$ is obtained by projecting $d\,b$ upon $A\,B$ at $b_1$, drawing $b_1\,z$ at $60°$ to $A\,B$ and terminating it at $s_2$ by $C\,D$, drawn at the same height ($\frac{6}{10}''$) as before. A parallel to $b\,d$ through $s_2$ intersects $v\,s_1$ at $s$, giving one corner of the plan of the upper base, from which the rectangle $s\,t\,u\,v$ is completed, with sides parallel to those of the lower base.

Fig. 240.



As the recess has vertical sides we may draw its plan, $o\,p\,q\,r$, directly from the given dimensions, and show the depth by short-dash lines in each of the elevations.

The ordinary elevations are derived from the plan as in preceding problems; that is, for the front elevation, $a'\,u'\,s'\,d'$, by verticals through the plans, terminating according to their height, either on $a'\,d'$ or on $u'\,s'$, $\frac{6}{10}''$ above it. For the side elevation, $e''\,v''\,t''\,b''$, with the *heights* as in the front elevation, the distances to the right or left of $s''$ equal those of the plans of the same points from $s\,i$, regarding the latter as the h. t. of a central, vertical plane, parallel to V.

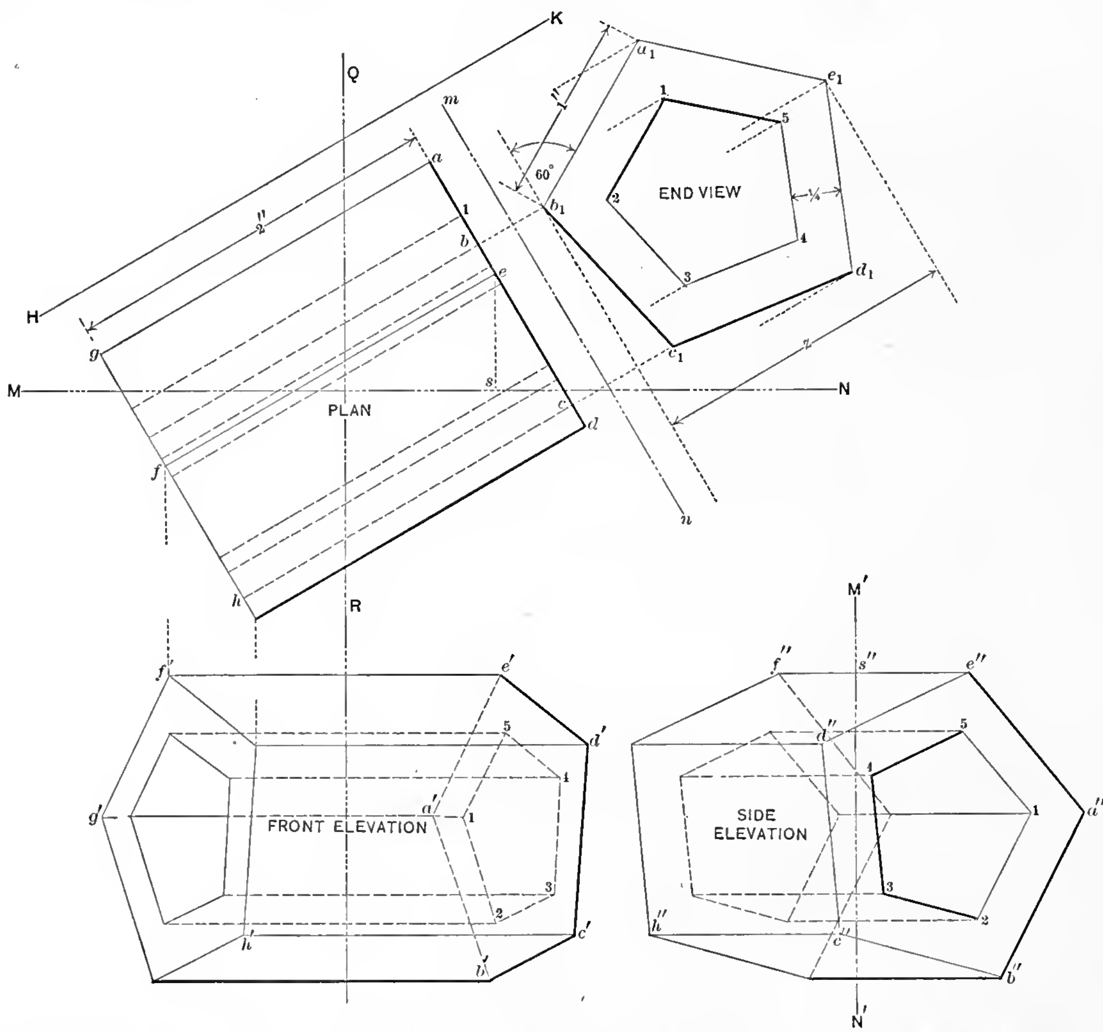
The plane $S\,T$ of *right section, perpendicular to the axis* $K\,L$, cuts the block in a section whose true size is shown in the line-tinted figure $g_1\,h_1\,k_1\,l_1$, and whose construction hardly needs detailed treatment after what has preceded. The shaded, longitudinal section, on central, vertical plane $K\,L$, also interprets itself by means of the lettering.

*The true size of any face*, as $a\,u\,v\,e$, may be shown by rabatment about a horizontal edge, as $a\,e$. As $v$ is actually $\frac{6}{10}''$ above the level of $e$, we see that $v\,e$ (in space) is the hypothenuse of a triangle of base $v\,e$ and altitude $\frac{6}{10}''$. Construct such a triangle, $v\,v_2\,e$, and with its hypothenuse $v_2\,e$ as a radius, and $e$ as a centre, obtain $v_1$ on a perpendicular to $a\,e$ through $v$ and representing the path of rotation. Finding $u_1$ similarly we have $a\,u_1\,v_1\,e$ as the actual size of the face in question.

If more views were needed than are shown the student ought to have no difficulty in their construction, as no new principles would be involved.

399. *To draw a hollow, pentagonal prism*, 2″ long; *edges* to be horizontal and inclined 35° to V; *base*, a regular pentagon of 1″ sides; one *face* of the prism to be inclined 60° to H; distance between inner and outer faces, $\frac{1}{4}''$.

Fig. 241.




In Fig. 241 let $H\,K$ be parallel to the plans of the axis and edges; it will make 35° with a horizontal line. Perpendicular to $H\,K$ draw $m\,n$ as the h. t. of an auxiliary, vertical plane, upon which we may suppose the base of the prism projected. In end view all the *faces* of the prism would be seen as *lines*, and all the *edges* as *points*. Draw $a_1\,b_1$, one inch long and at 60° to $m\,n$, to represent the face whose inclination is assigned. Completing the inner and outer pentagons, allowing $\frac{1}{4}''$ for the distance between faces, we have the end view complete. The plan is then

obtained by drawing parallels to $HK$ through all the vertices of the end view, and terminating all by vertical planes, $a\,d$ and $g\,h$, parallel to $m\,n$ and $2''$ apart.

The elevations will be included between horizontal lines whose distance apart is the extreme height $z$ of the end view; and all points of the *front elevation* are on verticals through their plans, and at heights derived from the end view. The most expeditious method of working is to draw a horizontal reference line, like that of Fig. 243, which shall contain the lowest edge of each elevation; measuring upward from this line lay off, on some random, vertical line, the distance of each point of the end view from a line (as the parallel to $m\,n$ through $b_1$ in Fig. 241, or $x\,y$ in Fig. 243) which represents the intersection of the plane of the end view by a horizontal plane containing the lowest point or edge of the object; horizontal lines, through the points of division thus obtained, will contain the projections of the corners of the front elevation, which may then be definitely located by vertical lines let fall from the plans of the same points. For example, $e'$ and $f'$, Fig. 241, are at a height, $z$, above the lowest line of the elevation, equal to the distance of $e_1$ from the dotted line through $b_1$; or, referring to Fig. 243, which, owing to its greater complexity, has its construction given more in detail, the distance upward from $M$ to line $G$ is equal to $g_1 g_2$ on the end view; from $M$ to $Q$ equals $q_1 q_2$, and similarly for the rest.

Since the profile plane is omitted in Fig. 241 we take $M'N'$ to represent the trace upon it of the auxiliary, central, vertical plane whose h. t. is $MN$; as already explained, all points of the *side elevation* are then at the same level as in the front elevation, and at distances to the right or left of $M'N'$ equal to the perpendicular distances of their *plans* from $MN$. For example, $e''s''$ equals $e\,s$.

The shade lines are located on the end view on the assumption that the observer is looking toward it in the direction $HK$.

400. *Projections of a hollow, pentagonal prism, cut by a vertical plane oblique to* V. Letting the data for the prism be the same as in the last problem, we are to find what modification in the appearance of the elevations would result from cutting through the object by a vertical plane $PQ$ (Fig. 242) and removing the part $h\,x\,d\,i$ which lies in front of the plane of section.

Each vertex of the section is on an edge of the elevation and is vertically below the point where $PQ$ cuts the plan of the same edge; the student can, therefore, readily convert the elevations of Fig. 241 into reproductions of those of Fig. 242 by drawing across the plan of Fig. 241 a trace $PQ$, similarly situated to the $PQ$ of Fig. 242. Supposing that done, refer in what follows to both Figures 241 and 242.

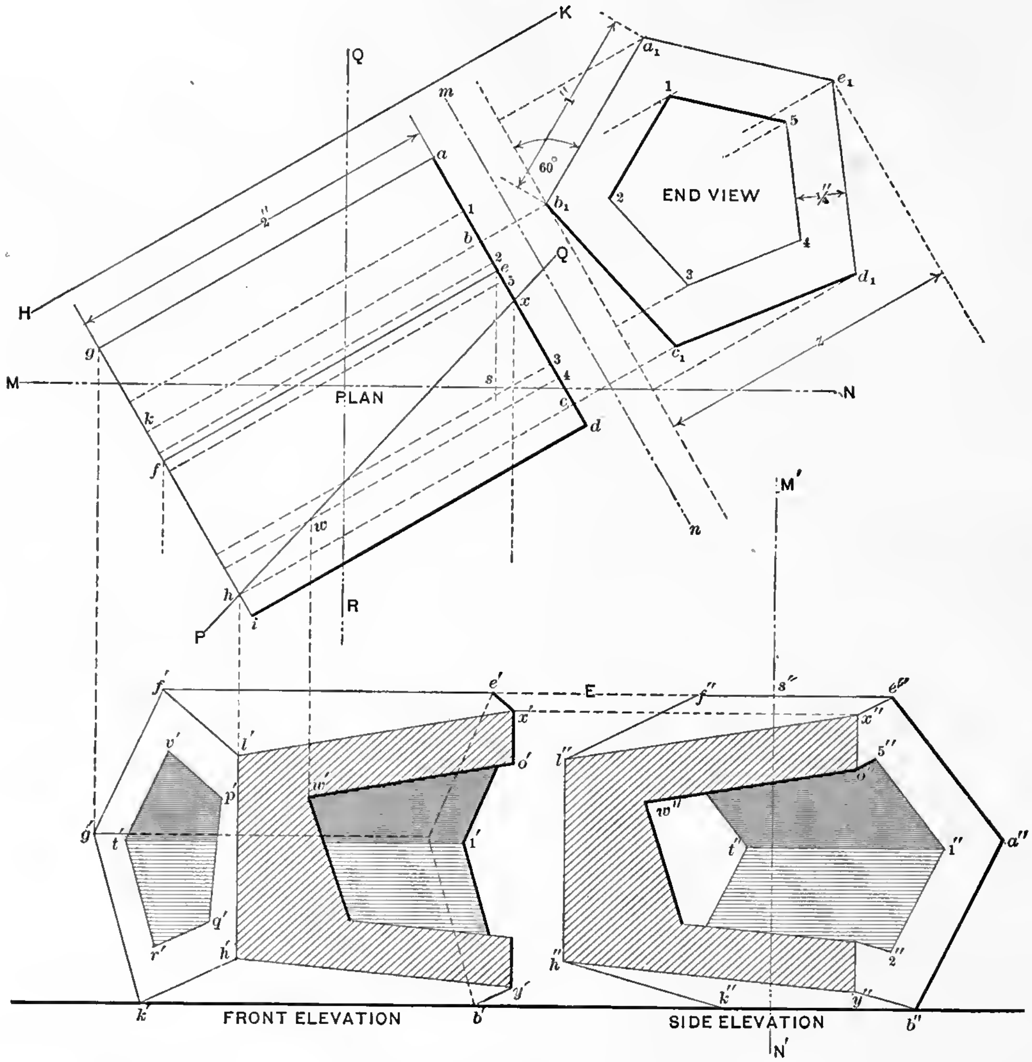
Since $PQ$ contains $h$ we find $h'$ as one corner of the section. Both *ends* of the prism being *vertical*, they will be cut by the vertical plane $PQ$ in vertical lines; therefore $h'l'$ is vertical until the top of the prism is reached, at $l'$. Join $l'$ with $x'$, the latter on the vertical through $x$—the intersection of $PQ$ with the right-hand top edge $e\,d$, $e'd'$. From $x$ the cut is vertical until the interior of the prism is reached, at $o'$, on the line 5–4. We next reach $w'$ on edge No. 4. The line $o'w'$ has to be parallel to $x'l'$ (two parallel planes are cut by a third in parallel lines); but from $w'$ the interior edge of the section is not parallel to $l'h'$, since $PQ$ is not cutting a vertical *end*, but the inclined, interior surface. The other points hardly need detailed description, being similarly found.

The side elevation is obtained in accordance with the principle fully described in Art. 396 and summarized in Art. 397 (b). $M'N'$ represents the same plane as $MN$; $e''s''$ equals $e\,s$, and analogously for other points.

401. In his elementary work in projections and sections of solids the student is recommended to lay an even tint of burnt sienna, medium tone, over the projections of the object, after which

any *section* may be line-tinted; and, if he desires to further improve the appearance of the views, distinctions may be made between the tones of the various surfaces by overlaying the burnt sienna with flat or graded washes of India ink.

Fig. 242.




402. *Projections of an L-shaped block, after being cut by a plane oblique to both* V *and* H; *the block also to be inclined to* V *and* H, *and to have running through it two, non-communicating, rectangular openings, whose directions are mutually perpendicular.*

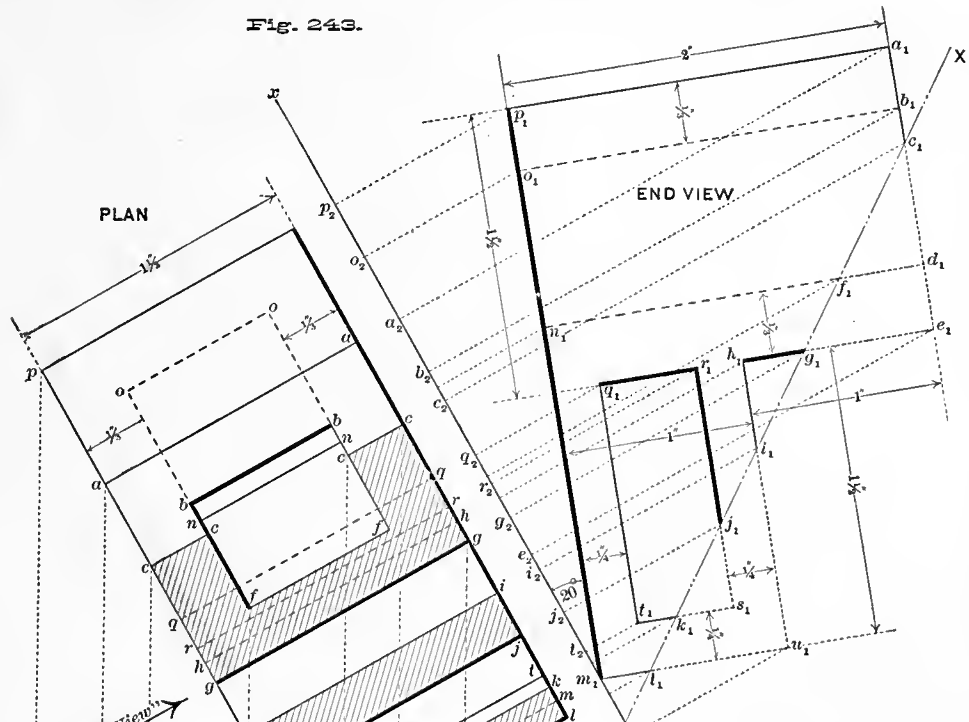
If the dotted lines are taken into account the front elevation in Fig. 243 gives a clear idea of the shape of the original solid. The end view and plan give the dimensions.

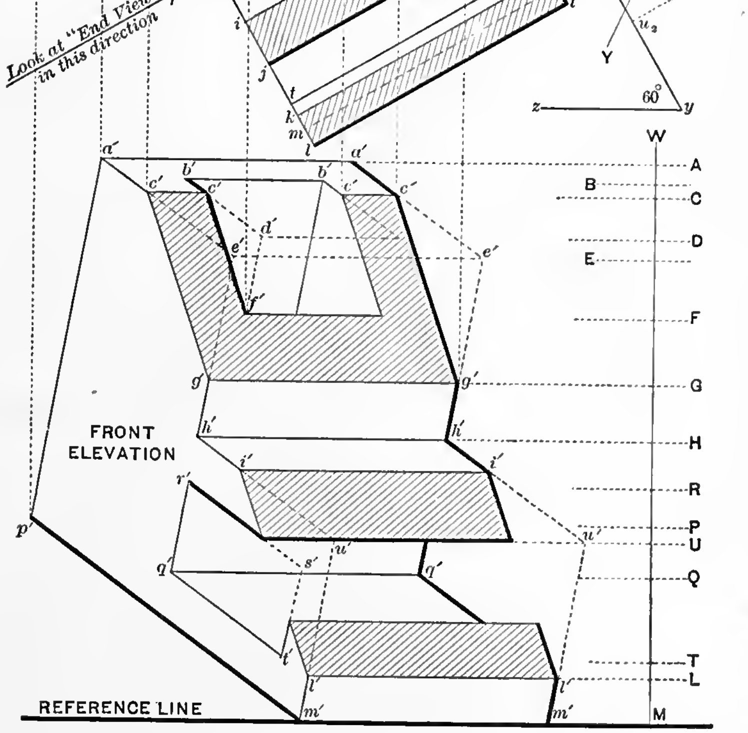
Requiring the horizontal edges of the block to be inclined 30° to V, draw the first line $xy$ at 60° to the horizontal; the *plans* of all the horizontal edges will be perpendicular to $xy$.

Let the inclination of the bottom of the block to H be 20°. This is shown in the end view by drawing $m_1p_1$ at 20° to $xy$. All the edges of the end view of the object will then be parallel or perpendicular to $m_1p_1$ and should be next drawn to the given dimensions.

The central opening, $b_1 d_1 n_1 o_1$, through the larger part of the block, has its faces all $\frac{1}{3}''$ from the outer faces. In the plan this is shown by drawing the lines lettered $of$ at a distance of $\frac{1}{3}''$ from the boundary lines, which last are indicated as $1\frac{1}{2}''$ apart.

Fig. 243.

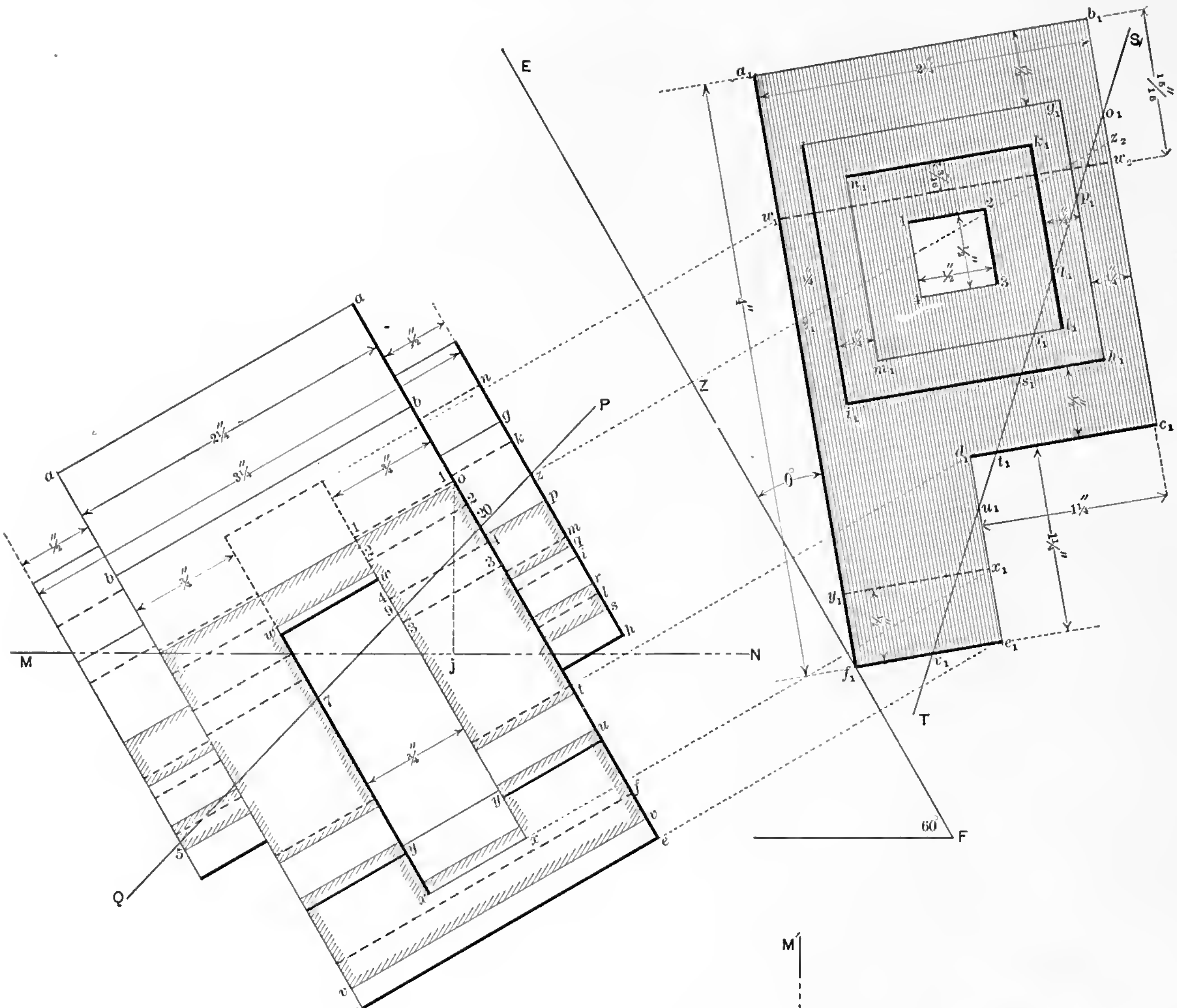


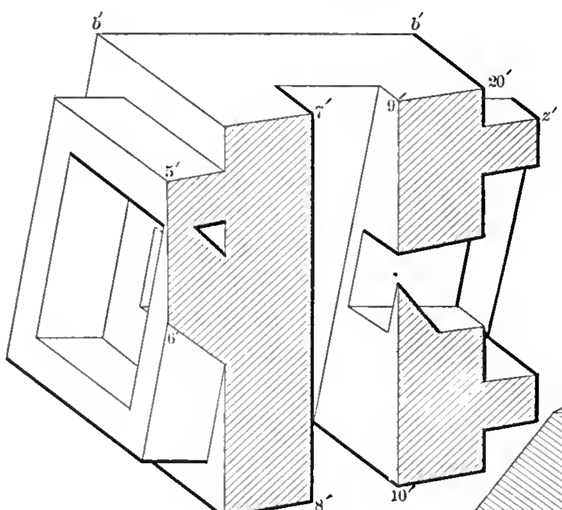




The opening $q_1 r_1 s_1 t_1$ has three of its faces $\frac{1}{4}''$ from the outer surfaces of the block, while the fourth, $q_1 r_1$, is in the same plane as the outer face $h_1 e_1$.

The cutting plane $XY$ gives a section which is seen in end view in the lines $c_1 g_1$, $i_1 j_1$ and $k_1 l_1$; while in plan the section is projected in the shaded portion, obtained, like all other parts of the plan, by perpendiculars to $xy$ from all the points of the end view.

For the front elevation draw first the "reference line." To provide against overlapping of projections the reference line should be at a greater distance below the lowest point, $l$, of the plan, than the greatest height $(a_1 a_2)$ of the end view above $xy$. Then on $MW$ lay off from $M$ the heights of the various horizontal edges of the block, deriving them from the end view. Thus $a_1 a_2$ is the height of $Aa'$ from $M$; from $M$ to level $B$ equals $b_1 b_2$, etc. Next project to the level $A$ from points $aa$ of the plan, getting edge $a'a'$ of the elevation, and similarly for all the other corners of the block. *Notice that all lines that are parallel on the object will be parallel in each projection* (except when their projections coincide); also that *in the case of sections, those outlines will be parallel which are the intersection of parallel planes by a third plane.* These principles may be advantageously employed as checks on the accuracy of the construction by points. The construction of the side elevation is left to the student.

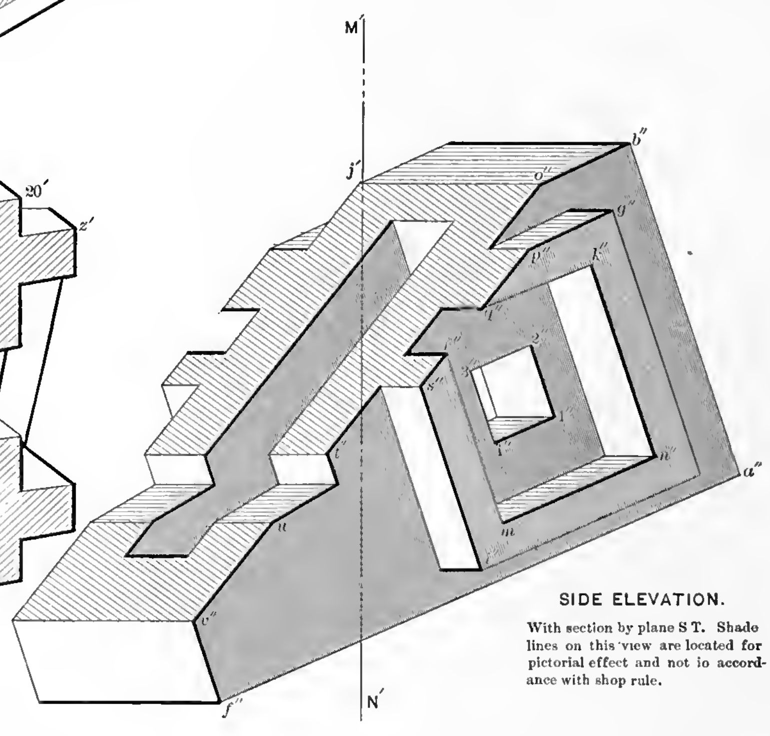


FRONT ELEVATION.

With section made by vertical plane P Q

E′ Reference line F′




SIDE ELEVATION.

With section by plane S T. Shade lines on this view are located for pictorial effect and not in accordance with shop rule.

403. *Projections and sections of a block of irregular form, with two mutually perpendicular openings through it, and with equal, square frames projecting from each side.*

In Fig. 244 the side elevation shows clearly the object dealt with, while we look to the end view for most of the dimensions. The large central opening extends from $w_1 w_2$ to $x_1 y_1$. The width of the main portion of the block is shown in plan as $2\frac{1}{4}''$, between the lines lettered $a\,e$. The square frames project $\frac{1}{2}''$ from the sides, while the width of the central opening between the lines $w\,x$ is $\frac{3}{4}''$.

Two section planes are indicated, $S\,T$ across the end view, and $P\,Q$—a vertical plane—across the plan; the section made by plane $S\,T$ is, however, shown only in fringed outline on the plan, though fully represented on the side elevation. The front elevation shows the section made by plane $P\,Q$, with the visible portion of that part of the object that is behind the cutting plane.

Although detailed explanation of this problem is unnecessary after what has preceded, yet a brief recapitulation of the various steps in the construction of the views may be appreciated by some, before passing on to a more advanced topic.

(a) $E\,F$, the first line to draw, is the trace of the vertical plane on which the end view is projected, and is at an angle of $60°$ to a horizontal line in order that the edges of the object (as $a\,a$, $b\,b$ - - - - $e\,e$) may be inclined at $30°$ to the front vertical plane, which we may assume as one of the conditions of the problem.

(b) A rotation of the object through an angle $\theta°$ about a horizontal axis that is perpendicular to $E\,F$, as, for example, the edge through $f$, is shown by the inclination of the end view to $E\,F$ at an angle $a_1 f_1 E = \theta°$.

(c) Drawing the end view at the required angle to $E\,F$ we next derive the plan therefrom by perpendiculars to $E\,F$, terminating them on parallels to $E\,F$ (as the lines $a\,e$, $w\,x$, $n\,h$, etc.,) whose distances apart conform to given data.

(d) *The elevations.* For these a common reference line $E'\,F'\,f''$ is taken, horizontal, and sufficiently below the plan to avoid an overlapping of views.

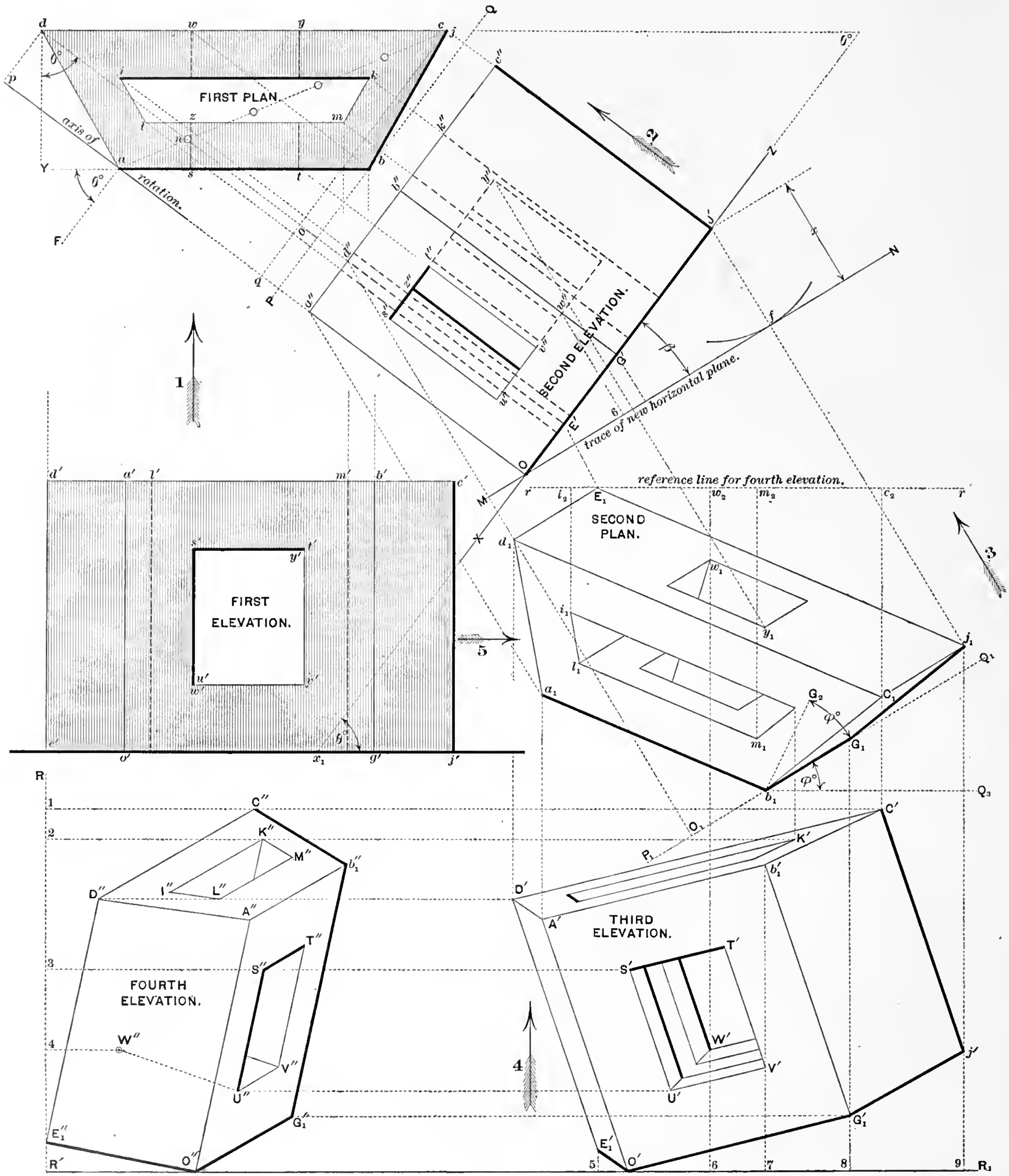
For the *front elevation* any point as $b'$, is found vertically below its plan $b$, and is as far from $E'\,F'$ as $b_1$ is—perpendicularly—from $E\,F$.

The height at which the section plane $P\,Q$ cuts *any* line is similarly obtained. Thus at $z$ it cuts the vertical end face of the block in a line which is carried over on the end view in the indefinite line $Z z_2$; the portions of $Z z_2$ which lie on the end view of the *frame* $g_1 h_1 i_1$ are the only real parts to transfer to the front elevation, and are seen on the latter, vertically below $z$ and running from $z'$ down; their distances from $E'\,F'$ being simply those from $Z$, on the end view, transferred.

*The side elevation.* Any point or edge is at the same *level* on the side elevation as on the front; hence the edge through $b''$ is on $b'\,b'$ produced. The distances to the right or left of $M'\,N$ equal those of the corresponding points on the plan from $M\,N$; thus $o''j'$ equals $o\,j$, etc.

404. *Changed planes of projections.* In the problems of Arts. 399–403 the employment of an "end view"—which was simply an *auxiliary elevation*—has prepared the student for the further use of planes other than the usual planes of projection; and if the *auxiliary plan* is now mastered he is prepared to deal with any case of rotation of object about vertical or horizontal axes, since new and properly located planes of projection are their practical equivalent.

In Fig. 245 the object is represented in its initial position by the line-tinted figures marked "first plan" and "first elevation." The third and fourth elevations show somewhat more pictorially that it is a hollow, truncated, triangular prism, having through it a rectangular opening that is perpendicular to the front and rear faces.


FIRST PLAN.
axis of rotation.
SECOND ELEVATION.
trace of new horizontal plane.
FIRST ELEVATION.
reference line for fourth elevation.
SECOND PLAN.
FOURTH ELEVATION.
THIRD ELEVATION.

(a) *Rotation about a vertical axis,* or its equivalent, *a change in the vertical plane of projection.* Result: *second elevation* derived from *first plan and elevation.*

Let the axis be one of the vertical edges of the object, as that at $d$ in the first plan; also let the rotation be through an angle $Y d o$ or $\theta°$, ($\theta$ being taken, for convenience, equal to the angle $Y a F$, which—with the line $p q$—will be employed in a later construction). If we were actually to rotate the object through an angle $\theta$ the new *plan* would be the exact counterpart of the first, but its horizontal edges would make an angle $\theta$ with their former direction, and the new *elevation* would partly overlap the first one. To avoid the latter unnecessary complication, as also the duplication of the plan, we make the first plan do double duty, since we can accomplish the equivalent of rotation of the object by taking a *new vertical plane* that makes an angle $\theta$ with the plane on which the first elevation was made. This equivalence will be more evident if some small object, as a piece of india rubber, is placed on the "first plan" with its longer edges parallel to $a b$, and is then viewed in the direction of arrow No. 2 through a pane of glass standing vertically on $X Z$; after which turn both the object and the glass through the angle $\theta$ until the glass stands vertically on $e' j'$ and then view in the direction of arrow No. 1.

The second plane may be located *anywhere,* as long as the angle $\theta$ is preserved; $X Z$, making angle $\theta$ at $x_1$ with $e' j'$, is, therefore, a *random position* of the new plane, and the projection upon it is our "second elevation."

Since the *heights* of the various corners of an object remain unchanged during rotation about a vertical axis we will find all points of the second elevation at distances from the reference line $X Z$ that are derived from the first elevation, and laid off on lines drawn perpendicular to $X Z$ from the vertices of the plan: thus $a O$ is perpendicular to $X Z$, and $O a''$ equals $o' a'$; $c'' J'$ equals $c' j'$, etc.

(b) *Rotation about a horizontal axis,* or its equivalent, *the adoption of a new horizontal plane.* Result: *second plan* derived from *first plan and second elevation.*

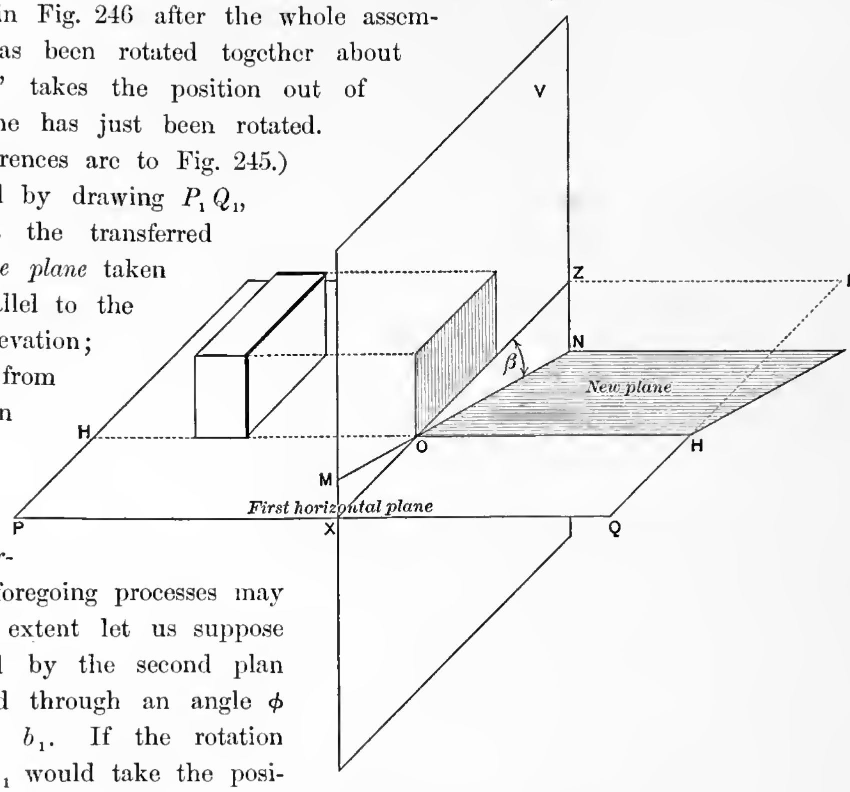
Having in the last case illustrated the method of complying with the condition that rotation should occur *through a given angle* (which is incidentally shown again, however, in the next construction) we now choose an axis $p q$ so as to illustrate a different kind of requirement, viz.: that during rotation the *heights* of any two points of the object, which were at first *at the same level as the axis,* shall be in some predetermined ratio, regardless of the *amount* of rotation. In the figure it is assumed that $e'(d)$ is to be at one-fifth the height of $j' j$, and that rotation shall occur about an axis passing through the lower end $o'$ of the vertical edge at $a$. By drawing $a d$ and $a j$, dividing the latter into five equal parts, and joining $d$ with $n$—the first point of division from $a$—we obtain the direction $d n$, parallel to which the axis $p q$ is drawn through $a$. The distance $d p$ is then one-fifth of $j q$, and they shorten in the same ratio, as rotation occurs.

After locating the axis the next step is, invariably, the drawing of an elevation upon a plane perpendicular to the axis. This we happen, however, to have already in our "second elevation," having, in the interest of compactness, so taken $\theta$ in the preceding case that the vertical plane $X Z$ would be perpendicular to the axis we are now ready to use.

Any rotation of the object about $p q$ will, evidently, not change the *form* of the "second elevation" but simply incline it to $X Z$. But, as before, instead of actually rotating the object, which would probably give projections overlapping those from which we are working, we adopt a new plane $M N$ as a horizontal plane of projection, so taken that it fulfills either of the following conditions: (a) that the object should be rotated about $p q$ through an angle $J' O N = \beta$; (b) that the corner $J'$ should be higher than $O$ by an amount $x$, $M N$ being drawn tangent to an arc having $J'$ for its centre, and $J' f$ (equal to $x$) for its radius.

Reference to Fig. 246 may make it clearer to some that $MN$ is the trace of the new plane upon the vertical plane whose h. t. is $XZ$; that $ON$ lies vertically below the line $XZ$ and is as truly perpendicular to the axis of rotation as is $XZ$; also that in Fig. 245 a view in the direction of arrow No. 3 (i. e., perpendicular to $MN$) is equivalent to a view perpendicular to the plane V in Fig. 246 after the whole assemblage of planes and object has been rotated together about $HOH$ until the "new plane" takes the position out of which the first horizontal plane has just been rotated.

Fig. 246.




(The remainder of the references are to Fig. 245.)

*The second plan* is obtained by drawing $P_1Q_1$, parallel to $MN$, to represent the transferred trace $PQ$ of a vertical *reference plane* taken through some edge $b$ and parallel to the plane $ZON$ of the second elevation; then any point $d_1$ is as far from $P_1Q_1$ as the same point $d$ on the first plan is from $PQ$; similarly, from point $w_1$ to $P_1Q_1$ equals distance $wb$.

(c) *Further rotation about vertical axes.* To show how the foregoing processes may be duplicated to any desired extent let us suppose that the object, as represented by the second plan and elevation, is to be rotated through an angle $\phi$ about a vertical line through $b_1$. If the rotation actually occurred, the plan $b_1G_1$ would take the position $b_1G_2$, and the other lines of the plan would take corresponding positions in relation to a vertical plane on $P_1Q_1$. A new vertical plane on $b_1Q_3$, at an angle $\phi$ to $b_1G_1$, will, however, evidently hold the same relation to the plan as it stands, and transferring such new plane forward to $O'R_1$ we then obtain the points of the new (third) elevation by letting fall perpendiculars to $O'R_1$ from the vertices of the second plan, and on them laying off heights above $O'R_1$ equal to those of the same points above $MN$ in the *second elevation.* Thus $j'9$ equals $J'f$; $W'6$ in the fourth equals $W''6$ in the second.

*The fourth elevation* is a view in the direction of arrow No. 5, giving the equivalent of a ninety-degree rotation of the object from its last position. To obtain it take a reference line $rr$ through some point of the second plan, and parallel to $O'R_1$; then $RR'$ represents the vertical plane on $rr$, transferred. From $RR'$ lay off—on the *levels* of the same points in the third elevation—distance $1C'' = c_2C_1$; $4W'' = w_2w_1$, as in preceding analogous constructions.

## THE DEVELOPMENT OF SURFACES.

405. *The development of surfaces* is a topic not altogether new to the student who has read Chapter V and the earlier articles of this chapter;* so far, however, it has occurred only incidentally, but its importance necessitates the following more formal treatment, which naturally precedes problems on the *interpenetration of surfaces,* of which a "development" is usually the practical outcome.

* The following articles should be carefully reviewed at this point: 120; 191; 344–6; 389, and Case 6 of Art. 396.

A *development* of a surface, using the term in a practical sense, is a piece of cardboard or, more generally, of sheet-metal, of such shape that it can be either directly *rolled up* or *folded* into a model of the surface. Mathematically, it would be the *contact-area*, were the surface rolled out or unfolded upon a plane.

The "shop" terms for a developed surface are "surface in the flat," "stretch-out," "roll-out"; also, among sheet-metal workers it is called a *pattern;* but as *pattern-making* is so generally understood to relate to the patterns for castings in a foundry, it is best to employ the qualifying words *sheet-metal* when desiring to avoid any possible ambiguity.

406. The mathematical nature of the surfaces that are capable of development has been already discussed in Arts. 344–346. Those most frequently occurring in engineering and architectural work are the right and oblique forms of the pyramid, prism, cone and cylinder.

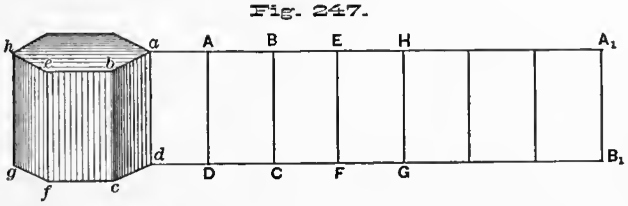
407. In Art. 120 *the development of a right cylinder* is shown to be a *rectangle* of base equal to $2\pi r$ and altitude $h$, where $h$ is the height of the cylinder and $r$ is the radius of its base.

408. *The development of a right cone* is proved, by Art. 191, to be a *circular sector*, of radius equal to the slant height $R$ of the cone, and whose angle $\theta$ is found by means of the proportion $R : r :: 360° : \theta$; $r$ being the radius of the base of the cone.

409. *The development of a right pyramid* is illustrated in Art. 389, and in Case 6 of Art. 396.

410. We next take up *right* and *oblique prisms*, and the *oblique pyramid*, *cone* and *cylinder;* while for the sake of completeness, and departing in some degree from what was the plan of this work when Arts. 345 and 346 were written, the *regular solids* will receive further treatment, and also the *developable helicoid*.

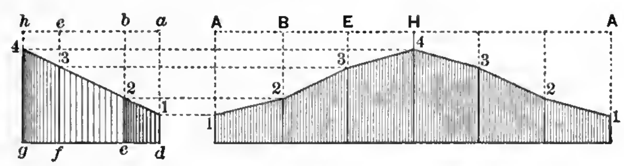
Fig. 247.




411. *The development of a right prism.* Fig. 247 represents a regular, hexagonal prism. The six faces being equal, and $c\,b\,c\,f$ showing their actual size, we make the rectangles $A\,B\,C\,D$, $B\,E\,F\,C$, etc., each equal to $c\,b\,c\,f$; then $A\,A_1$ equals the perimeter of the upper base, and we have the rectangle $A\,A_1\,B_1\,D$ for the development sought.

412. *The development of a right prism below a cutting plane.* Taking the same prism as in the last article develop first as if there were no section to be taken into account. This gives, as before, a rectangle of length $A\,A_1$ and of altitude $a\,d$, divided into six equal parts. Then project, from each point where the plane cuts an edge, to the same edge as seen on the development.

Fig. 248.

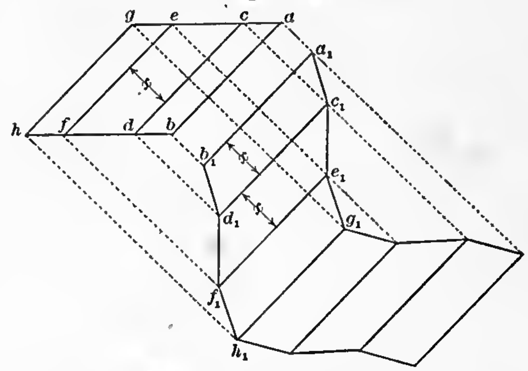



413. *Right section. Rectified curve. Developed curve.* A plane perpendicular to the axis of a surface cuts the latter in a *right section.* The bases of right cones, pyramids, cylinders and prisms fulfil this condition and require no special construction for their determination; but the development of an oblique form usually involves the construction of a right section and then the laying off on a straight line of a length equal to the perimeter of such section. Should the right section be a *curve* its equivalent length on a straight line is called its *rectification*, which should not be confounded with its *development*, the latter not being necessarily straight.

414. *The development of an oblique prism, when the faces are equal in width.* In Fig. 249 an oblique, hexagonal prism is shown, with $x$ for the width of its faces. Since the perimeter of a right section would evidently equal $6x$ we may directly lay off $x$ six times on some perpendicular

to the edges, as that through $a$. The seven parallels to $a\,b$, drawn at distances $x$ apart, will contain the various edges of the prism as it is rolled out on the plane; and the positions of the extremities are found by perpendiculars from their original positions. The initial position $a_1 b_1$ is *parallel to* but at any distance from $a\,b$. The base edges are evidently unequal.

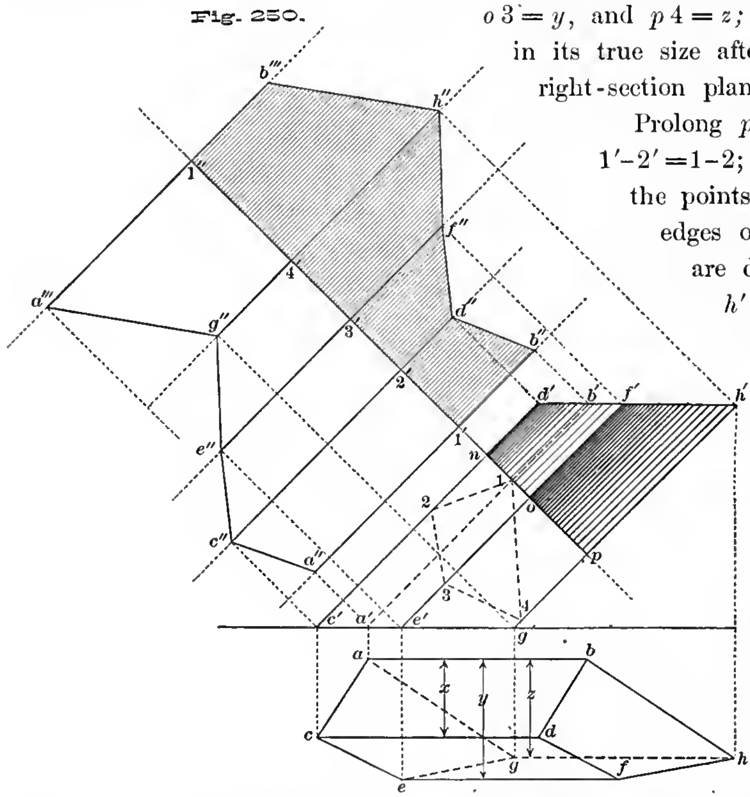
Fig. 249.



415. *The development of an oblique prism whose faces are unequal in width.*

In Fig. 250 $c'd'h'g'$ is the elevation of the prism; $n\,p$ a plane of right section. To get 1–2–3–4, the true shape of the right section, we require $a\,b\,h\,f\,e$, the plan of the prism.* Assuming that to have been given imagine next a vertical reference plane standing on $a\,b$. The right section plane $n\,p$ cuts the edge $c'd'$ at $n$, which is at a distance $x$ in front of the assumed reference plane. Make $n\,2 = x$. Similarly make $o\,3 = y$, and $p\,4 = z$; then 1–2–3–4 is the right section, seen in its true size after being revolved about the trace of the right-section plane upon the assumed reference plane.

Fig. 250.



Prolong $p\,n$ indefinitely, and on its extension make $1'–2' = 1–2$; $2'–3' = 2–3$, etc. Parallels to $c'd'$ through the points of division thus obtained will contain the edges of the developed prism, and their lengths are definitely determined by perpendiculars, as $h'h''$, $f'f''$, from the extremities of the original edges.

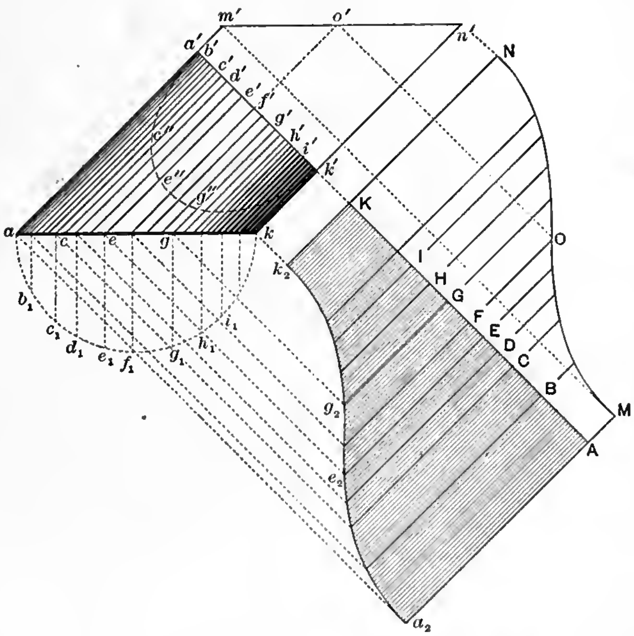
416. *The development of an oblique cylinder, having a circular base and elliptical right section.* Let $a\,m'n'k$, Fig. 251, be an oblique cylinder with circular base. Take any plane of right section, as $a'k'$. Draw various elements, as those through $b'$, $c'$, etc., and from their lower extremities erect perpendiculars to $a\,k$, as $c\,c_1$, terminating them on the arc $a\,f_1\,k$, which represents the half base of the cylinder. On $c\,c'$ make $c'c'' = c\,c_1$; on $e\,e'$ take $e'e'' = e\,e_1$, and similarly obtain other points on the elements, through which the curve $a'c''e''g''k'$ can be drawn, this being one-half of the curve of right section, shown after revolution about its shorter diameter. Making $K\,A$ equal to the rectified semi-ellipse just obtained, lay off $A\,C =$ arc $a'c''$; $C\,E =$ arc $c''e''$, etc., and through the points of division thus obtained on $K\,A$ draw indefinite parallels to the axis of the cylinder. These will represent the elements on the development, and are limited by the dotted lines drawn perpendicular to the original elements and through their extremities.

The area $a_2 k_2 N M$ is the development of one-half of the cylinder, the shaded area representing all between $a'k'$ and the base $a\,k$.

* In the interest of compactness the "First Angle" position of the views (Art. 385) is employed in Figs. 250, 253 and 255.

417. *The development of an oblique pyramid.* The development will evidently consist of a series of triangles having a common vertex. To ascertain the length of any edge we may carry it *into* or *parallel to* a plane of projection. Thus in Fig. 252 the edge $v\,b$ is carried into the vertical plane at $v\,b''$. Its true length is the hypothenuse of a right-angled triangle of base $o\,b = o\,b''$, and altitude $v\,o$.

Fig. 251.



In Fig. 253 a pyramid is shown in plan and elevation. Making $o\,a'' = v\,a$ we have $v'\,a''$ for the actual length of edge $v'\,a'$, a construction in strict analogy to that of Fig. 252. The plan $v\,b$ being parallel to the base line shows that $v'\,b'$ is the actual length of that edge. By carrying $v\,c$ to $v\,e_1$, where it becomes parallel to V, and then projecting $e_1$ to $c''$ we get $v'\,e''$ for the true length of edge $v'\,e'$.

Fig. 252.

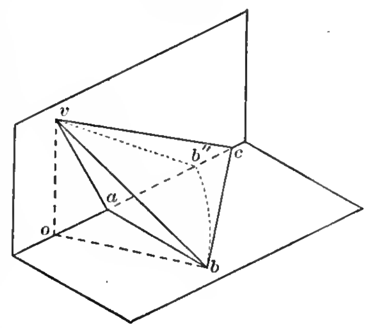
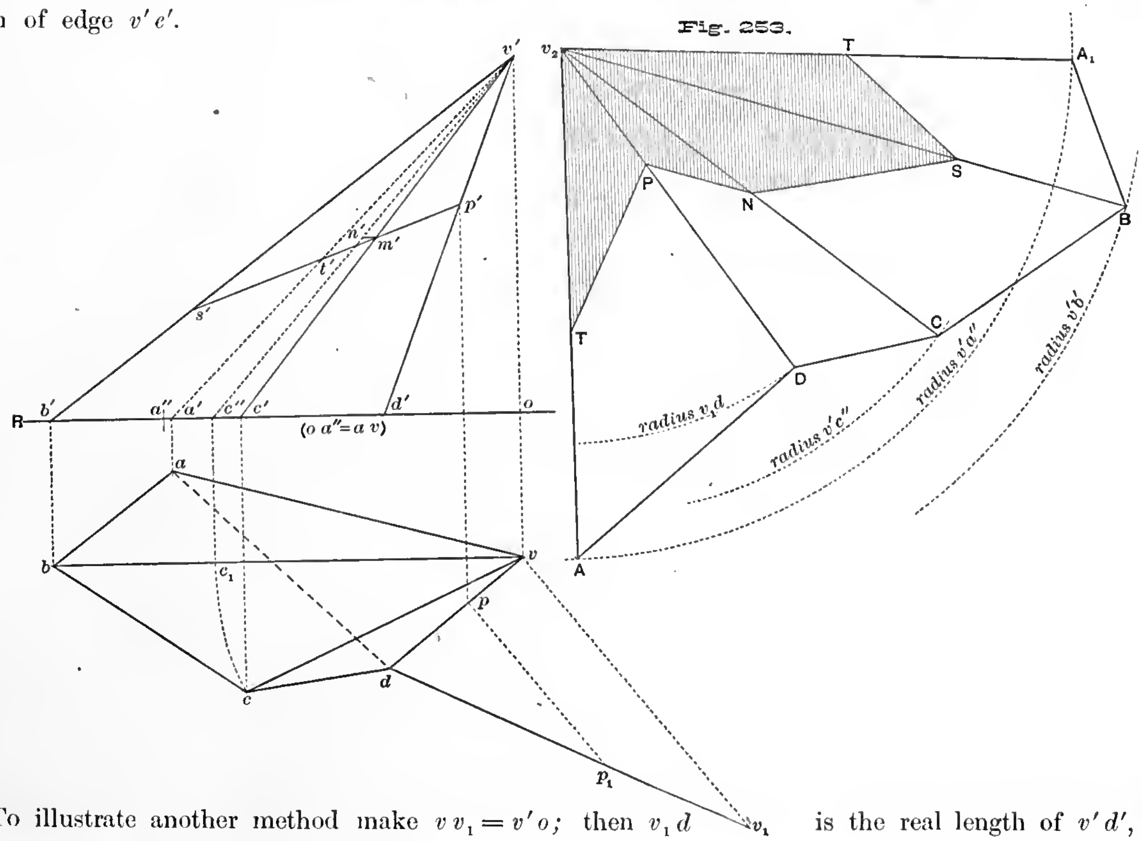


Fig. 253.



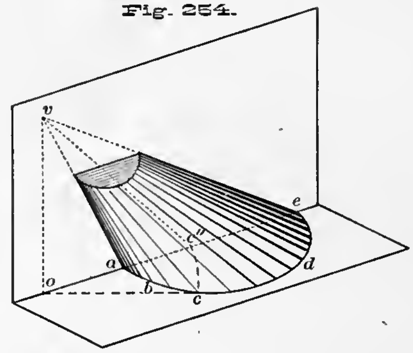

To illustrate another method make $v\,v_1 = v'\,o$; then $v_1\,d$ is the real length of $v'\,d'$, shown by rabatment into H.

*For the development* take some point $v_2$ and from it as a centre draw arcs having for radii the ascertained lengths of the edges. Thus, letting $v_2A$ represent the initial edge of the development, take $A$ as a centre, $a\,d$ as a radius, and cut the arc of radius $v_1d$ at $D$; then $A\,v_2D$ is the *development* of the face $a\,v\,d$, $a'v'd'$. With centre $D$ and radius $d\,c$ obtain $C$ on the arc of radius $v'c''$; similarly for the remaining faces, completing the development $v_2 - A\,D \ldots A_1$.

The shaded area $v_2 - T\,P \ldots T$ is the development of that part of the pyramid above the oblique plane $s'p'$, found by laying off, on the various edges as seen in the development, the distances along those edges from the vertex to the cutting plane; thus $v_2N = v'n'$, the real length of $v'm'$; $v_2P = v_1p_1$; the length of $v'p'$; $v_2S = v's'$, the only elevation showing actual length.

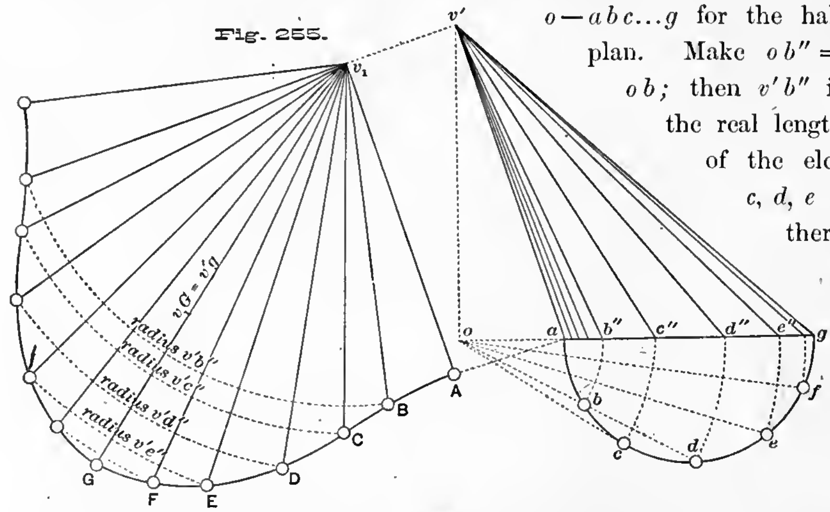
418. *The development of an oblique cone.* The usual method of solving this problem gives a result which, although not mathematically exact, is a sufficiently close approximation for all practical purposes. In it the cone is treated as if it were a pyramid of many sides. The length of any element is then found as in the last problem. Thus in Fig. 254 an element $v\,c$ is carried to $v\,c''$ about the vertical axis $v\,o$.

Fig. 254.




In Fig. 255 we have $v'.a\,g$ for the elevation of the cone, and $o - a\,b\,c \ldots g$ for the half plan. Make $o\,b'' = o\,b$; then $v'b''$ is the real length of the element whose plan is $o\,b$. Similarly, $c$, $d$, $e$ and $f$ are carried by arcs to $a\,g$ and there joined with $v'$.

Fig. 255.

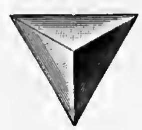



For the development make $v_1A$ equal and parallel to $v'a$, and at any distance from it. With $v_1$ as a centre draw arcs with radii equal to the true lengths of the elements; then, as in the pyramid, make $A\,B = arc\;\;a\,b$; $B\,C = arc\;\;b\,c$, etc.

The greater the number of divisions on the semi-circle $a\,b \ldots g$ the more closely will the development approximate to theoretical exactness.

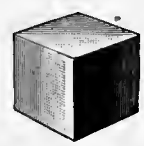
419. *The five regular convex solids*, with the forms of their developments, are illustrated in Figs. 256–265. They have already been defined in Art. 345, and that five is their limit as to number is thus shown: The faces are to be equal, regular polygons, and the sum of the plane angles forming a solid angle must be less than four right angles; now as the angles of *equilateral triangles* are 60° we may evidently have groups of three, four or five and not exceed the limit; with *squares* there can be groups of three only, each 90°; with regular *pentagons*, their interior angles being 108°, groups of three; while hexagons are evidently impracticable, since three of their interior angles would exactly equal four right angles, adapting them perfectly—and only—to plane surfaces. (See Fig. 131.)

Fig. 256.



TETRAHEDRON.

Fig. 257.



CUBE.

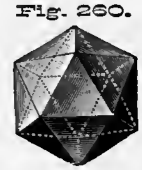
Fig. 258.

DODECAHEDRON.

Fig. 259.

OCTAHEDRON.

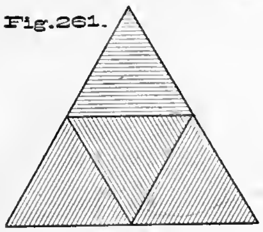
Fig. 260.



ICOSAHEDRON.

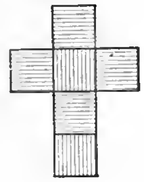
The dihedral angles between the adjacent faces of regular solids are as follows: 70° 31′ 44″ for the tetrahedron; 90° for the cube; 109° 28′ 16″ for the octahedron; 116° 35′ 54″ for the dodecahedron; and 138° 11′ 23″ for the icosahedron.

Fig. 261.



TETRAHEDRON.

Fig. 262.



CUBE.

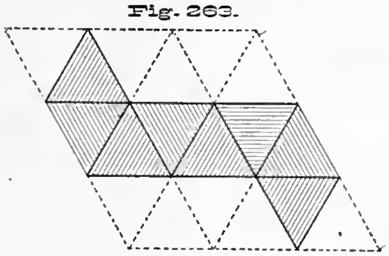
A sphere can be inscribed in each regular solid and can also as readily be circumscribed about it.

The relation between $d$, the diameter of a sphere, and $e$, the edge of an inscribed regular solid, is illustrated graphically by Fig. 266, but may be otherwise expressed as follows:

For the *tetrahedron* $d : e :: \sqrt{3} : \sqrt{2}$; for the *cube* $d : e :: \sqrt{3} : 1$

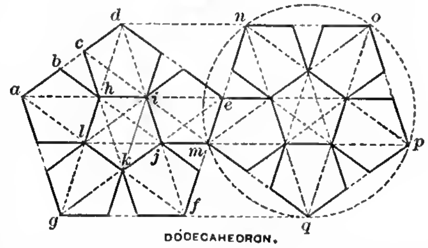
" " *octahedron* $d : e :: \sqrt{2} : 1$; " " *dodecahedron* $e =$ the greater segment of the edge of an inscribed cube when the latter has been medially divided, that is, in extreme and mean ratio.

Fig. 263.



OCTAHEDRON.

Fig. 264.

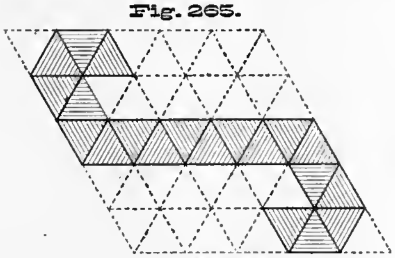


DODECAHEDRON.

For the *icosahedron* $e =$ the chord of the arc whose tangent is $d$; i. e., the chord of 63° 26′ 6″.

Reference to Figs. 256–260 and the use of a set of cardboard models which can readily be made by means of Figs. 261–265 will enable the student to verify the following statements as to those ordinary views whose construction would naturally precede the solution of problems relating to these surfaces.

Fig. 265.

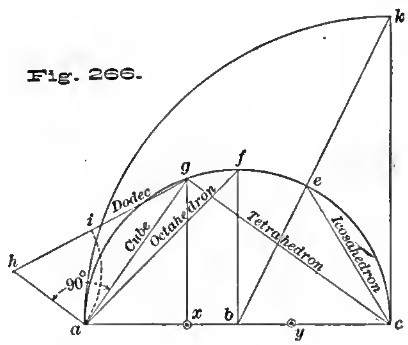


ICOSAHEDRON.

In all but the tetrahedron each face has an equal, opposite, parallel face, and except in the cube such faces have their angular points alternating. (See Figs. 260, 267, 268.)

The *tetrahedron* projects as in Fig. 256, upon a plane that is parallel to either face.

Fig. 266.




The *cube* projects in a square upon a plane parallel to a face, while on a plane perpendicular to a body diagonal it projects as a regular hexagon, with lines joining three alternate vertices with the centre.

The *octahedron*, which is practically two equal square pyramids with a common base, projects in a square and its diagonals, upon a plane perpendicular to either body diagonal; in a rhombus and shorter diagonal when the plane is parallel to one body diagonal and at 45° with the other

Fig. 267.

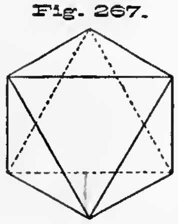


Fig. 268.

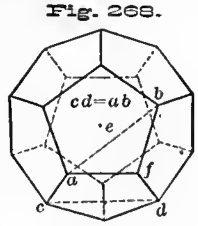
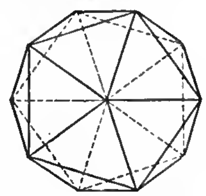



Fig. 269.



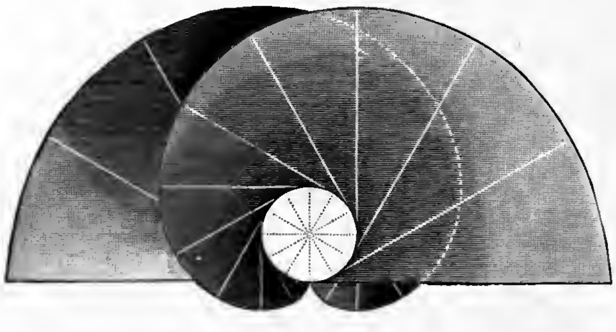
two; and (as in Fig. 267) in a regular hexagon with inscribed triangles (one dotted) when it is projected upon a plane parallel to a face.

The *dodecahedron* projects as in Fig. 268 whenever the plane of projection is parallel to a face.

Fig. 260 represents the *icosahedron* projected on a plane parallel to a face, and Fig. 269 when the projection-plane is perpendicular to an axis.

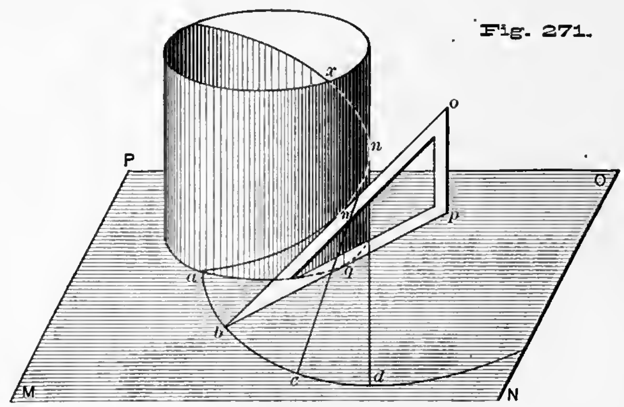
420. *The Developable Helicoid.* When the word *helicoid* is used without qualification it is understood to indicate one of the *warped* helicoids, such as is met with, for example, in screws, spiral staircases and screw propellers. There is, however, a *developable* helicoid, and to avoid confusing it with the others its characteristic property is always found in its name. As stated in Art. 346, it is generated by moving a straight line tangentially on the ordinary helix, which curve (Art. 120) cuts all the elements of a right cylinder at the same angle. Fig. 209 illustrates the completed surface pictorially; Fig. 270 shows one orthographic projection, and in Fig. 271 it is seen in process of generation by the hypothenuse of a right-angled triangle that rolls tangentially on a cylinder.

Fig. 270.



The construction just mentioned is based on the property of non-plane curves that at any point the curve and its tangent make the same angle with a given plane; if, therefore, the helix, beginning at $a$, crosses each element of the cylinder at an angle equal to $o\,b\,p$ in the rolling triangle, the hypothenuse of the latter will evidently move not only tangent to the cylinder, but also to the helix.

Fig. 271.




The following important properties are also illustrated by Fig. 271:

(a) The involute* of a helix and of its horizontal projection are identical, since the point $b$ is the extremity of both the rolling lines, $o\,b$ and $p\,b$.

(b) The length of any tangent, as $m\,b$, is that of the helical arc $m\,a$ on which it has rolled.

(c) The horizontal projection $b\,q$ of any tangent $b\,m$ equals the rectification of an arc $a\,q$ which is the *projection* of the helical arc from the initial point $a$ to the point of tangency $m$.

*The development of one nappe of a helicoid* is shown in Fig. 273. It is merely the area between a circle and its involute; but the radius $\rho$, of the base circle, equals $r \sec^2 \theta$,† in which $r$ is

---

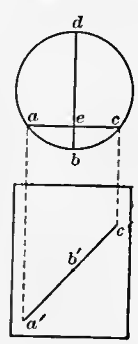
* For full treatment of the involute of a circle refer to Arts. 186 and 187.

† This relation is due to considerations of *curvature.* At *any* point of *any* curve its *curvature* is its rate of departure from its tangent at that point. Its *radius of curvature* is that of the *osculatory circle* at that point. (Art. 380.) Now from the nature of the two *uniform* motions imposed upon a point that generates a helix (Art. 120) the curvature of the latter must be *uniform;* and if developed upon a plane *by means of its curvature* it must become a circle—the only *plane* curve of *uniform curvature.* The radius of the developed helix will, obviously, be the radius of curvature of the space helix. Following Warren's method of proof in establishing its value let $a$, $b$ and $c$ (Fig. 272) be three equi-distant points on a helix, with $b$ on the foremost element; then $a'c'$ is the elevation of the circle containing these points. One diameter of the circle $a'b'c'$ is projected at $b'$. It is the hypothenuse of a right-angled triangle having the chord $b\,c$, $b'c'$, for its base. Let $2\rho$ be the diameter of the circle $a'b'c'$; $2r = b\,d$, that of the cylinder. Using capitals for points *in space* we have $\overline{BC}^2 = 2\rho \times b\,e$; also $\overline{bc}^2 = 2r \times b\,e$; whence, dividing like members and substituting trigonometric functions (see note p. 31), we have $\rho = r \sec^2 \beta$, in which $\beta$ is the angle between the line $BC$ and its projection.

Let $\theta$ be the inclination of the tangent to the helix at $b'$. If, now, both $A$ and $C$ approach $B$, the angle $\beta$ will approach $\theta$ as its limit; and when $A$, $B$ and $C$ become *consecutive* points we will have $\rho = r \sec^2 \theta =$ the radius of the osculatory circle $=$ *the radius of curvature.*

For another proof, involving the radius of curvature of an ellipse, see Olivier, *Cours de Géométrie Descriptive,* Third Ed., p. 197.

Fig. 272.



the radius of the cylinder on which the helix originally lay, and $\theta$ is the angle at which the helix crosses the elements. To determine $\rho$ draw on an elevation of the cylinder, as in Fig. 274, a line $ab$, tangent to the helix at its foremost point, as in that position its inclination $\theta$ is seen in actual size; then from $o$, where $ab$ crosses the extreme element, draw an indefinite line, $os$, parallel to $cd$, and cut it at $m$ by a line $am$ that is perpendicular to $ab$ at its intersection with the front element $ef$ of the cylinder; then $om = \rho = r\ sec^2\theta$. For we have $oa = on \sec\theta = r\sec\theta$; and $on\,(=r) : oa :: oa : om$; whence $om = r\sec^2\theta = \rho$.

Fig. 273.

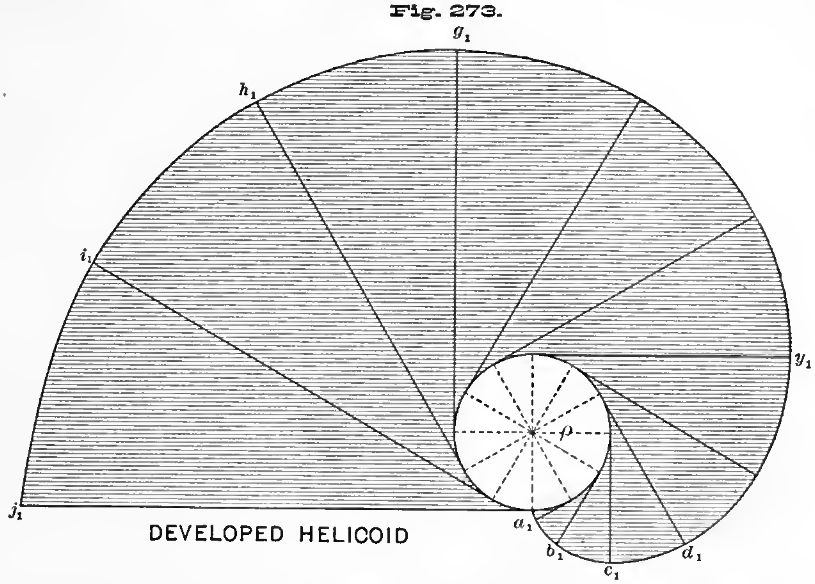
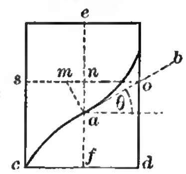



Fig. 274.




The circumference of circle $\rho$ equals $2\pi r\sec\theta$, the actual length of the helix, as may be seen by developing the cylinder on which the latter lies. The elements which were tangent to the helix maintain the same relation to the developed helix, and appear *in their true length* on the development.

The student can make a model of one nappe of this surface by wrapping a sheet of Bristol board, shaped like Fig. 273, upon a cylinder of radius $r$ in the equation $r\sec^2\theta = \rho$; or a two-napped helicoid by superposing two equal circular rings of paper, binding them on their inner edges with gummed paper, making one radial cut through both rings, and then twisting the inner edge into a helix.

## THE INTERSECTION OF SURFACES.

421. *When plane-sided surfaces intersect*, their outline of interpenetration is necessarily composed of straight lines; but these not being, in general, in one plane, form what is called a *twisted* or *warped polygon*; also called a *gauche polygon*.

422. If either of two intersecting surfaces is *curved* their common line will also be curved, except under special conditions.

423. When one of the surfaces is of uniform cross section—as a cylinder or a prism—its end view will show whether the surfaces intersect in a continuous line or in two separate ones. In Cases $a$, $b$, $c$, $d$ and $g$ of Fig. 275, where the end view of one surface either cuts but one limiting line of the other surface or is tangent to one or both of the outlines, the intersection will be a *continuous line*. Two separate curves of intersection will occur in the other possible cases, illustrated by $e$ and $f$, in which the end view of one surface either crosses both the outlines of the other or else lies wholly between them.

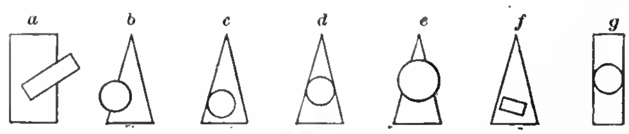


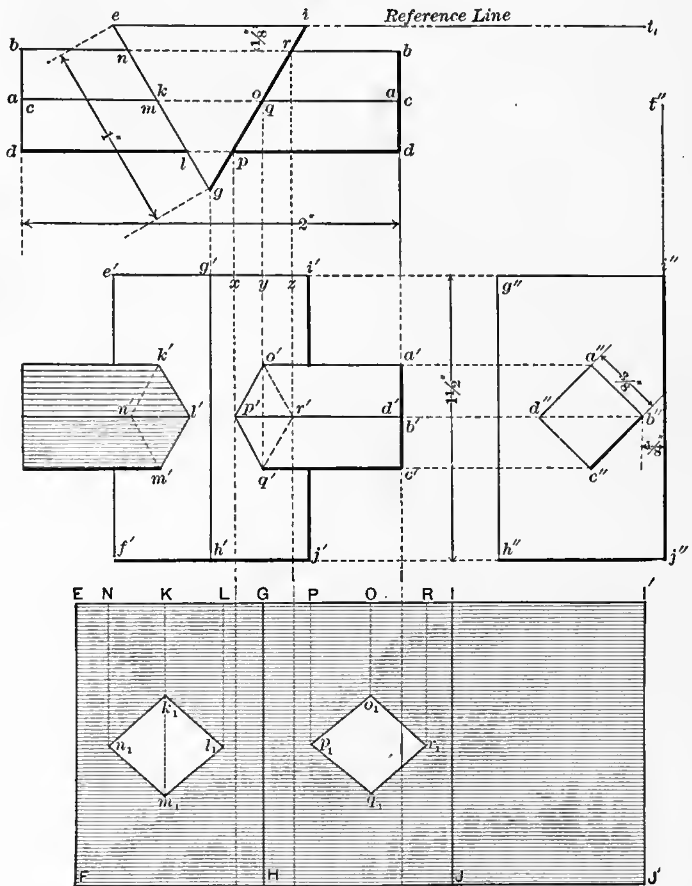

Fig. 275.

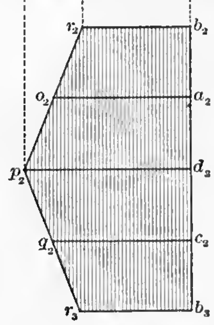
A cylinder will intersect a cone or another cylinder in a *plane curve* if its end view is tangent to the outlines of the other surface, as in $d$ and $g$, Fig. 275. Two cones may also intersect in a plane curve, but as the conditions to be met are not as readily illustrated they will be treated in a special problem. (See Art. 439).

424. In general, the line of intersection of two surfaces is obtained, as stated in Art. 379, by passing one or more auxiliary surfaces, usually *planes*, in such manner as to cut some easily constructed sections—as straight lines or circles—from each of the given surfaces; the meeting-points of the sections lying in any auxiliary surface will lie on the line sought.

The application of the principle just stated is much simplified whenever any face of either of the surfaces is so situated that it is projected in a *line*. This case is amply illustrated in the problems most immediately following.

Fig. 276.



The beginner will save much time if he will letter each projection of a point as soon as it is determined.

425. *The intersection of a vertical triangular prism by a horizontal square prism; also the developments.*

*The vertical prism* to be $1\frac{1}{2}''$ high and to have one face parallel to V; bases equilateral triangles of $1''$ side.

*The horizontal prism* to be $2''$ long, its basal edges $\frac{3}{8}''$, and its faces inclined 45° to H; its rear edges to be parallel to and $\frac{1}{8}''$ from the rear face of the horizontal prism.

The elevations of the axes to bisect each other.

Draw $e\,i$ horizontal and $1''$ long for the plan of the rear face of the vertical prism. Complete the equilateral triangle $e\,g\,i$ and project to levels $1\frac{1}{2}''$ apart, obtaining $e'f'$, $g'h'$, $i'j'$, on the elevation.

Construct an end view $g''i''j''h''$, using $t''j''$ to represent the reference line $e\,t$, transferred.

The end view of the horizontal prism is the square $a''b''c''d''$, having its diagonal horizontal and midway between the upper and lower bases of the other prism, and with its corner $b''$ one-eighth inch from $i''j''$. The plan and front elevation of the horizontal prism are next derived from the end view as in preceding constructions.

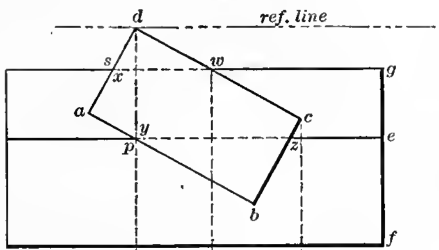
Since the lines $e\,g$ and $g\,i$ are the plans of vertical faces of one prism we note their intersection by the edges $a$, $b$, $c$ and $d$ of the horizontal prism—as at $n$, $m$, $l$, $p$, $q$, $r$—and project to the elevations of the same edges. Thus the edge $a\,a$ meets the other prism at $o$ and $k$, which project to the level of $a''$ at $o'$ and $k'$. Similarly for the remaining points.

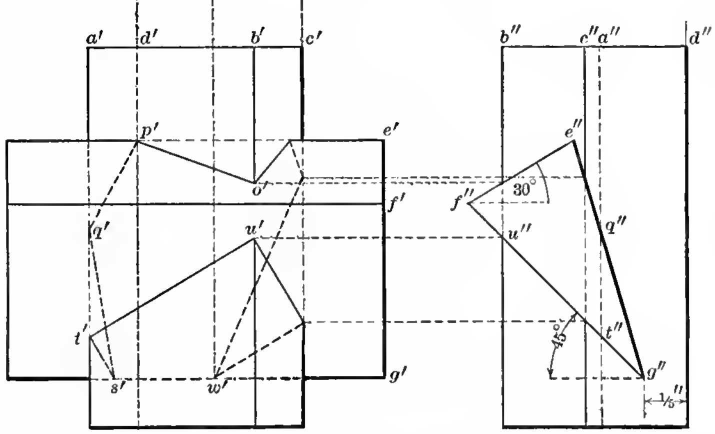
The development of the vertical prism is shown in the shaded rectangle $EJ'$, of length $3\,g\,i$ and altitude $e'f'$. (See Art. 411). The openings $o_1p_1q_1r_1$ and $k_1l_1m_1n_1$ are thus found: For $p_1$, which represents $p'$, make $GP = g\,p$, the true distance of $p'$ from $g'h'$; then $Pp_1 = x\,p'$. Similarly, $OG = o\,g$, and $Oq_1 = y\,q'$.

The right half of the horizontal prism, $o'\,a'\,c'\,q'$, is developed at $r_2\,b_2\,b_3\,r_3$ after the method of Art. 412.

426. *The intersection of two prisms, one vertical, the other horizontal, each having an edge exterior to the other.* The condition made will, as already stated (Art. 423), make the result a single *warped polygon.*

Fig. 277.





Let $a\,b\,c\,d$, $1'' \times \frac{1}{2}''$, be the plan of the vertical prism, which stands with its broader faces at some convenient angle $c\,w\,g$ to V. From it construct the front and side elevations, taking a reference plane through $d$ for the latter.

Let the horizontal prism be triangular (isosceles section) one face inclined 45° to H; another 30° to H; the rear edge to be $\frac{1}{5}''$ from that of the vertical prism.

Begin by locating $g''$ one-fifth inch from the right edge, draw $f''\,g''$ at 45°, making it of sufficient length to have $f''$ exterior to the other prism; then $f''\,e''$ at 30° to H, terminated at $e''$ by an arc of centre $g''$ and radius $g''\,f''$; finally $e''\,g''$. The edges $e'$, $f'$ and $g'$ of the front elevation are then projected from $e''$, $f''$ and $g''$.

The rear edge $g$ in the plan meets the face $a\,d$ at $s$, which projects to $s'$ on the elevation of the edge through $g'$.

Moving forward from $s$, the next edge reached, of either solid, is $a$, of the vertical prism. To ascertain the height at which it meets the other prism we look to the end view, finding $q''$ for the entrance and $t''$ for the exit. Being on the way up from $g''$ to $e''$ we use $q''$, reserving $t''$ until we deal with the face $g''\,f''$. Projecting $q''$ over to $q'$ on edge $a\,a'$ draw $q'\,s'$, dotted, since it is on a rear face.

Returning to $a$ and moving toward $b$ we next reach the edge $e$, whose intersection $p$ with $a\,b$ is then projected to edge $e'$ at $p'$ and joined with $q'$.

Fig. 278.

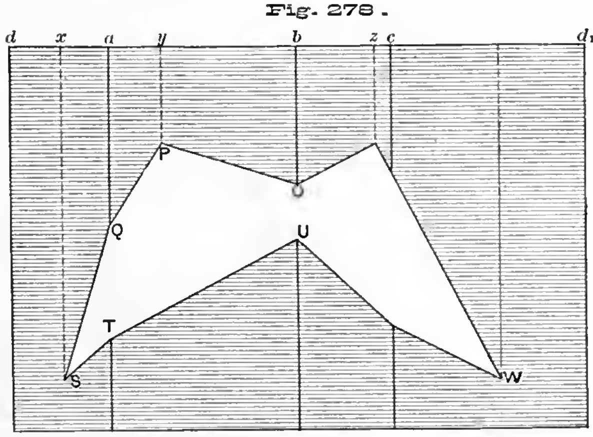
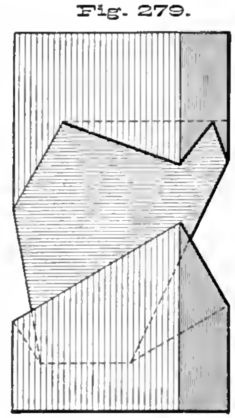


Fig. 279.



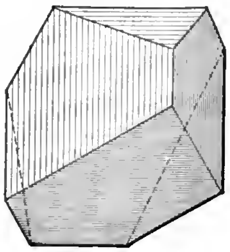
For the next edge, $b$, we obtain $o'$ from the side elevation, projecting from the intersection of $f''\,e''$ by the edge $b''$.

Moving from $b$ toward $w$, projecting to the front elevation from either the plan or the side elevation according as we are dealing with a horizontal edge or a vertical one, we complete the intersection.

The development of the vertical prism is shown in Fig. 278. As already fully described, $d\,d_1$ = perimeter $a\,b\,c\,d$ in Fig. 277; $a\,Q = a'\,q'$; $b\,O = b'\,o'$; $a\,x = a\,x$ (of Fig. 277); $x\,S$ = vertical distance of $s'$ from $a'\,c'$, etc.

Although not required in shop work the draughtsman will find it an interesting and valuable exercise to draw and shade either solid after the removal of the other; also to draw the common solid. The former is illustrated by Fig. 279; the latter by Fig. 280.

Fig. 280.



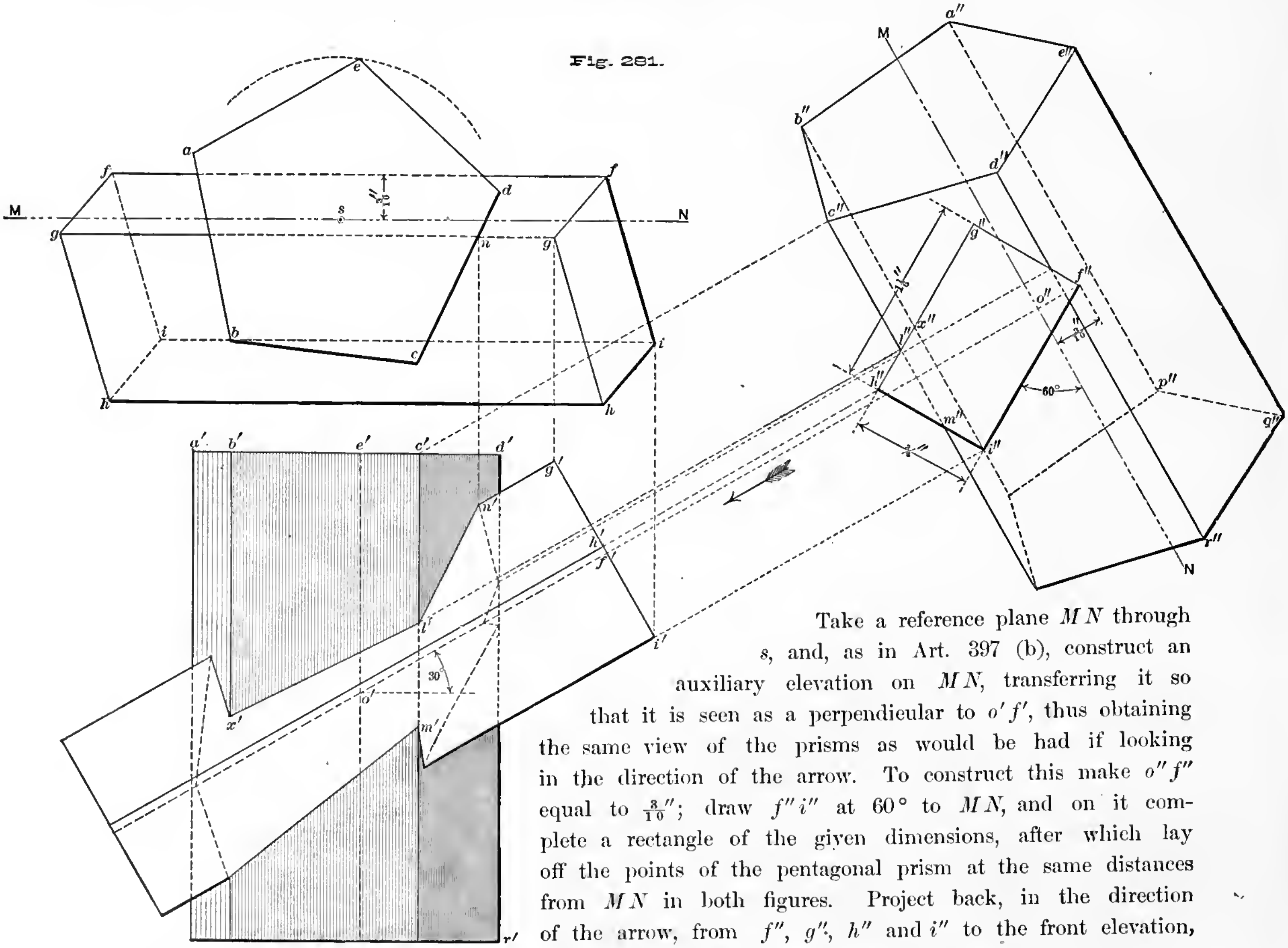
427. *The intersection of two prisms, one vertical, the other oblique but with edges parallel to V.*

Let $a\,b\,c\,d \ldots . a'r'$ (Fig. 281) be the plan and elevation of the vertical prism.

Let the oblique prism be (a) inclined 30° to H; (b) have its rear edge $\frac{3}{10}''$ back of the axis of the vertical prism; (c) have its faces inclined 60° and 30° respectively to V; (d) have a rectangular base $1\frac{1}{6}'' \times \frac{3}{4}''$. These conditions are fulfilled as follows:

Through some point $o'$ of the edge $e'o'$ draw an indefinite line, $o'f'$, at 30° to H, for the elevation of the rear edge, and $ff$, also indefinite in length at first but $\frac{3}{10}''$ back of $s$, for the plan.

Fig. 281.



Take a reference plane $MN$ through $s$, and, as in Art. 397 (b), construct an auxiliary elevation on $MN$, transferring it so that it is seen as a perpendicular to $o'f'$, thus obtaining the same view of the prisms as would be had if looking in the direction of the arrow. To construct this make $o''f''$ equal to $\frac{3}{10}''$; draw $f''i''$ at 60° to $MN$, and on it complete a rectangle of the given dimensions, after which lay off the points of the pentagonal prism at the same distances from $MN$ in both figures. Project back, in the direction of the arrow, from $f''$, $g''$, $h''$ and $i''$ to the front elevation, and draw $g'i'$ and the opposite base each perpendicular to $o'f'$ and at equal distances each side of $o'$.

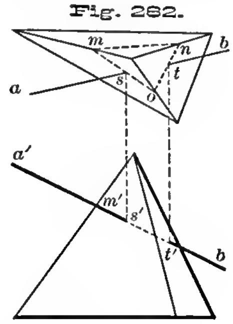
*For the intersection* we get any point $n'$ on an oblique edge, as $g'$, by noting and projecting from

$n$ where the plan $g\,g$ meets the face $c\,d$. For a vertical edge as $c'm'$ look to the auxiliary elevation of the same edge, as $c''$, getting $l''$ and $m''$ which then project back to $l'$ and $m'$.

The development need not again be described in detail but is left for the student to construct with the reminder that for the actual distance of any corner of the intersection from an edge of either prism he must look to that projection which shows the base of that prism in its true size: thus the distance of $l'$ from the edge $h'$ is $h''l''$.

428. *The intersection of pyramidal surfaces by lines and planes.* The principle on which the intersection of pyramidal surfaces by plane-sided or single curved surfaces would be obtained is illustrated by Figs. 282 and 283.

Fig. 282.

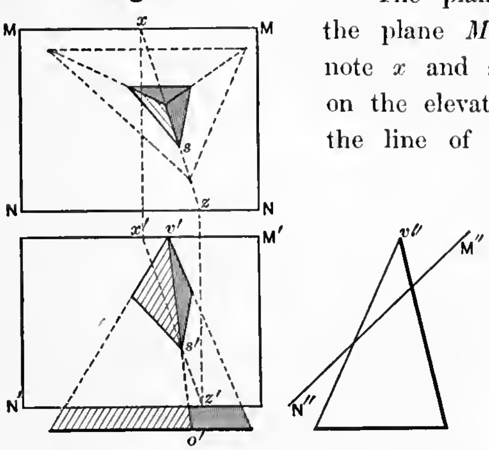



(a) In Fig. 282 the line $a\,b$, $a'b'$, is supposed to intersect the given pyramid. To ascertain its entrance and exit points we regard the elevation $a'b'$ as representing a *plane* perpendicular to V and cutting the edges of the pyramid. Project $m'$, where one edge is cut, to $m$, on the plan of the same edge. Obtaining $n$ and $o$ similarly we have $m\,n\,o$ as the plan of the section made by plane $a'b'$. The plan $a\,b$ meets $m\,n\,o$ at $s$ and $t$, the plans of the points sought, which then project back to $a'b'$ at $s'$ and $t'$ for the elevations.

As $a\,b$, $a'b'$, might be an edge of a pyramid or prism, or an element of a conical, cylindrical or warped surface, the method illustrated is of general applicability.

(b) In Fig. 283 the auxiliary planes are taken *vertical*, instead of perpendicular to V as in the last case.

Fig. 283.




The plane $MN$ cuts a pyramid. To find where any edge $v'o'$ pierces the plane $MN$ pass an auxiliary vertical plane $x\,z$ through the edge, and note $x$ and $z$, where it cuts the limits of $MN$; project these to $x'$ and $z'$ on the elevations of the same limits; draw $x'z'$, which is the elevation of the line of intersection of the original and auxiliary planes, and note $s'$, where it crosses $v'o'$. Project $s'$ back to $s$ on the plan of $v'o'$.

If a side elevation has been drawn, in which the plane in question is seen as a *line* $M''N''$, the *height* of the points of intersection can be obtained therefrom directly.

429. *The intersection of two quadrangular pyramids.* In Fig. 284 the pyramid $v\,.\,e\,f\,g\,h$ is vertical; altitude $v'z'$; base $e\,f\,g\,h$, having its longer edges inclined 30° to V.

*The oblique pyramid.* Let $s'y'$, the axis of the oblique pyramid, be parallel to V but inclined $\theta°$ to H, and be at some small distance (approximately $v\,k$) in front of the axis of the vertical pyramid; then $s\,c$ will contain the plan of the axis, and also of the diagonally opposite edges $s\,a$ and $s\,c$, if we make—as we may—the additional requirement that $a'c'$, the diagonal of the base, shall lie in the same vertical plane with the axis.

Instead of taking a separate end view of the oblique pyramid we may rotate its base on the diagonal $a'c'$ so that its foremost corner appears at $b''$ and the rear corner at $d''$, whence $b'$ and $d'$ are derived by perpendiculars $b''b'$ and $d''d'$, and then the edges $s'b'$ and $s'd'$. For the plans $b$ and $d$ use $s\,c$ as the trace of the usual reference plane, and offsets equal to $b'b''$ and $d'd''$, as previously.

The angle $a'c'd'$, or $\phi$, is the inclination of the shorter edges of the base to V.

*The intersection.* Without going into a detailed construction for each point of the outline of interpenetration it may be stated that each method of the preceding article is illustrated in this

problem, and that there is no special reason why either should have a preference in any case except where by properly choosing between them we may avoid the intersection of two lines at a very acute angle—a kind of intersection which is always undesirable.

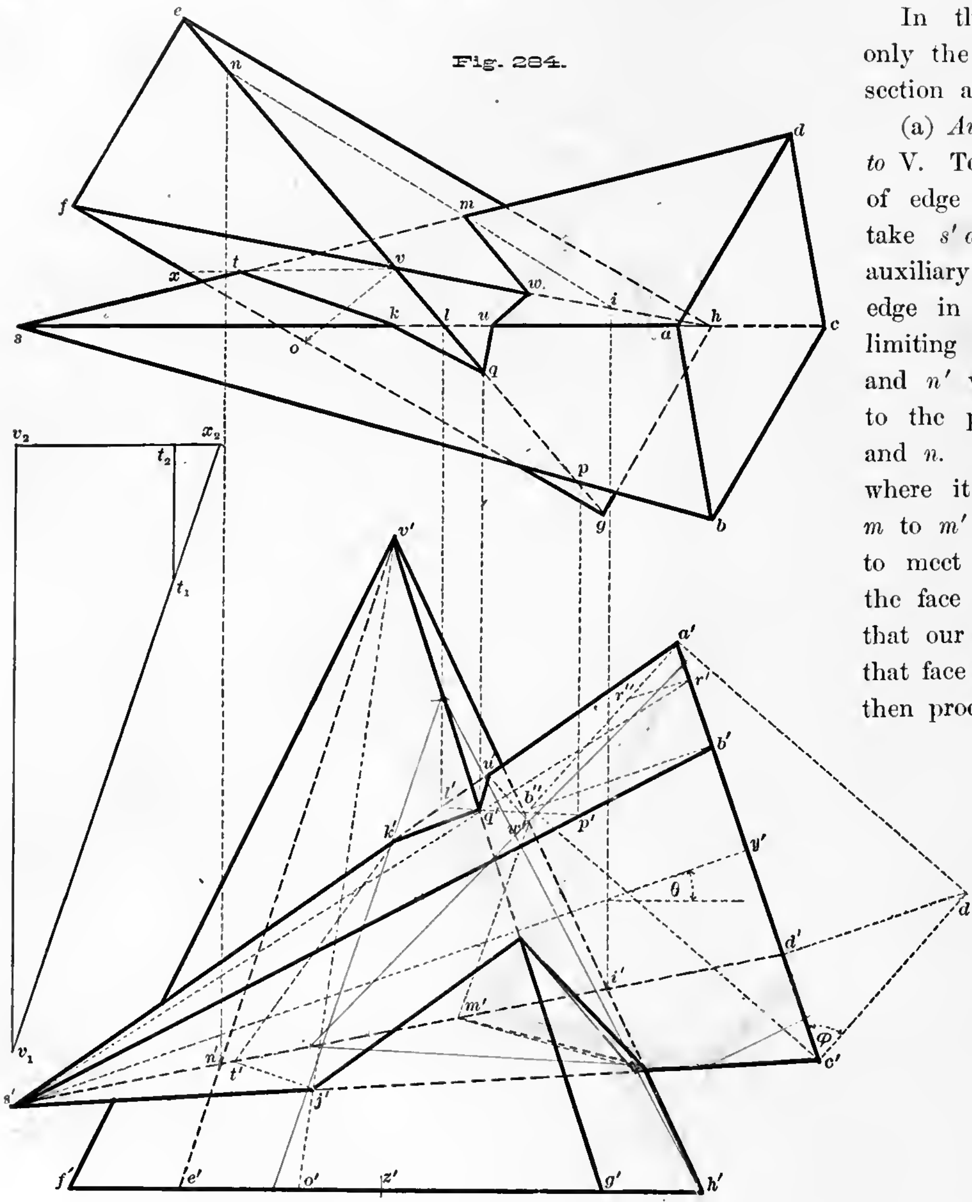


Fig. 284.

In the interest of clearness only the *visible* lines of the intersection are indicated on the plan.

(a) *Auxiliary plane perpendicular to* V. To find $m$, the intersection of edge $s\,d$ with the face $v\,h\,e$, take $s'\,d'$ as the trace of the auxiliary plane containing the edge in question; this cuts the limiting edges of the face at $i'$ and $n'$ which then project back to the plans of the edges at $i$ and $n$. Drawing $n\,i$ we note $m$, where it crosses $s\,d$, and project $m$ to $m'$ on $s'\,d'$. Had $n\,i$ failed to meet $s\,d$ within the limit of the face $v\,h\,e$ we would conclude that our assumption that $s\,d$ met that face was incorrect, and would then proceed to test it as to some other face, unless it was evident on inspection that the edge cleared the other solid entirely, as is the case with $s\,b$, $s'\,b'$, in the present instance. By using $s'\,b'$ as an auxiliary plane the student will obtain a graphic proof of failure to intersect.

(b) *Auxiliary plane vertical.* This case is illustrated by using $v\,g$ as the trace of an auxiliary vertical plane containing the edge $v\,g, v'\,g'$. Thinking this edge may possibly meet the face $s\,b\,a$ we proceed to test it on that assumption.

The plane $v\,g$ crosses $s\,a$ at $l$, and $s\,b$ at $p$; these project to $l'$ on $s'\,a'$ and to $p'$ on $s'\,b'$; then $p'\,l'$ meets $v'\,g'$ at $q'$, which is a real instead of an imaginary intersection since it lies between the actual limits of the face considered. From $q'$ a vertical to $v\,g$ gives $q$.

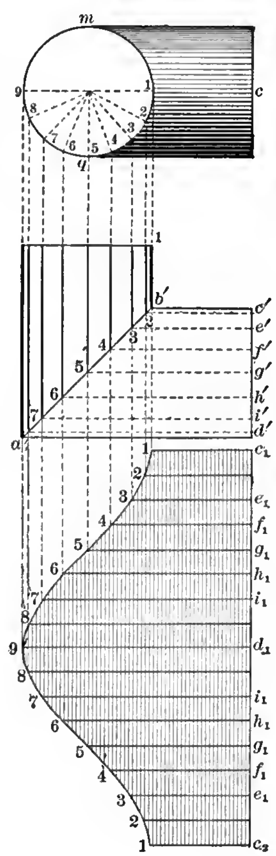
*The order of obtaining and connecting the points.* The start may be with *any* edge, but once under way the progress should be uniform, and each point joined with the preceding as soon as obtained. Two points are connected only when both lie on a single face of each pyramid.

Supposing that $q'$ was the point first found, a look at the plan would show that the edge $s\,a$ of the oblique pyramid would be reached before $v\,h$ on the other, and the next auxiliary plane would therefore be passed through $s\,a$ to find $u\,u'$; then would come $v\,h$ and $s\,d$. Running down from $m$ on the face $s\,d\,e$ we find the positions such that inspection will not avail, and the only thing to do is to try, at random, either a plane through $v\,h$ or one through $s\,e$; and so on for the remaining points.

*The developments.* No figure is furnished for these, as nearly all that the student requires for obtaining them has been set forth in Art. 396, Case 6. The only additional points to which attention need be called are the cases where the intersection falls on a *face* instead of an *edge*. For example, in developing the *vertical* pyramid we would find the development of $j'$ by drawing $v'\,j'$, prolonging it to $o'$, and projecting the latter to $o$, when $f\,x\,o$ would be the real distance to lay off from $f$ on the development of the base; then laying off *the real length* of $v'\,j'$ on $v'\,o'$ as seen in the development we would have the point sought. Similarly, for $t\,t'$, draw $v\,x$; make $v_2\,x_2 = v\,x$, and $v_2\,v_1 =$ altitude $v'\,z'$; then $v_1\,x_2$ is the true length of $v\,x$ (in space); also, making $v_2\,t_2 = v\,t$ and drawing $t_2\,t_1$, we find $v_1\,t_1$ to lay off in its proper place on the development of the same face $v\,f\,g$.

430. *An elbow or* T*-joint, the intersection of two equal cylinders whose axes meet.* Taking up curved surfaces the simplest case of intersection that can occur is the one under consideration, and which is illustrated by Fig. 285.

Fig. 285.

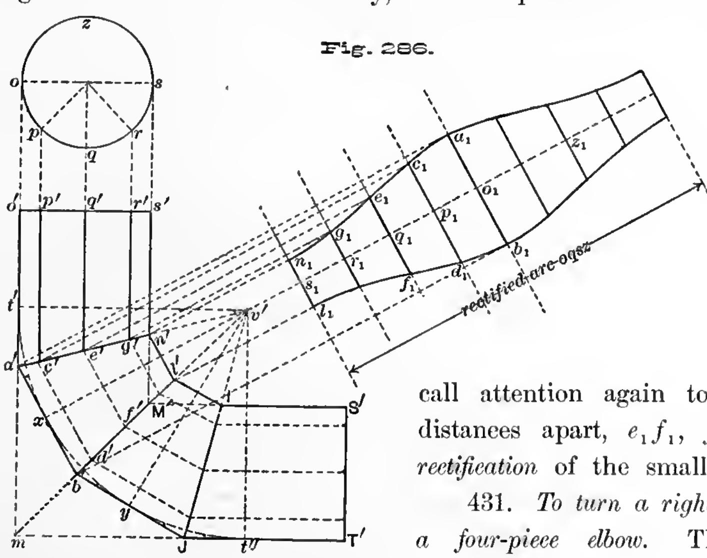


The conditions are those stated in Art. 423 for a *plane* intersection, which is seen in $a'\,b'$ and is actually an ellipse.

The vertical piece appears in plan as the circle $m\,q$. To lay off the *equidistant* elements on *each* cylinder it is only necessary to divide the half plan of one into equal arcs and project the points of division to the elevation in order to get the full elements, and where the latter meet $a'\,b'$ to draw the dotted elements on the other.

*The development* of the horizontal cylinder is shown in the line-tinted figure. The curved boundary, which represents the developed ellipse, is in reality a sinusoid. (Refer to Art. 171).

Fig. 286.




The relation of the developed elements to their originals, fully described in Art. 120, is so evident as to require no further remark, except to call attention again to the fact that their distances apart, $e_1\,f_1$, $f_1\,g_1$, etc., equal the *rectification* of the small arcs of the plan.

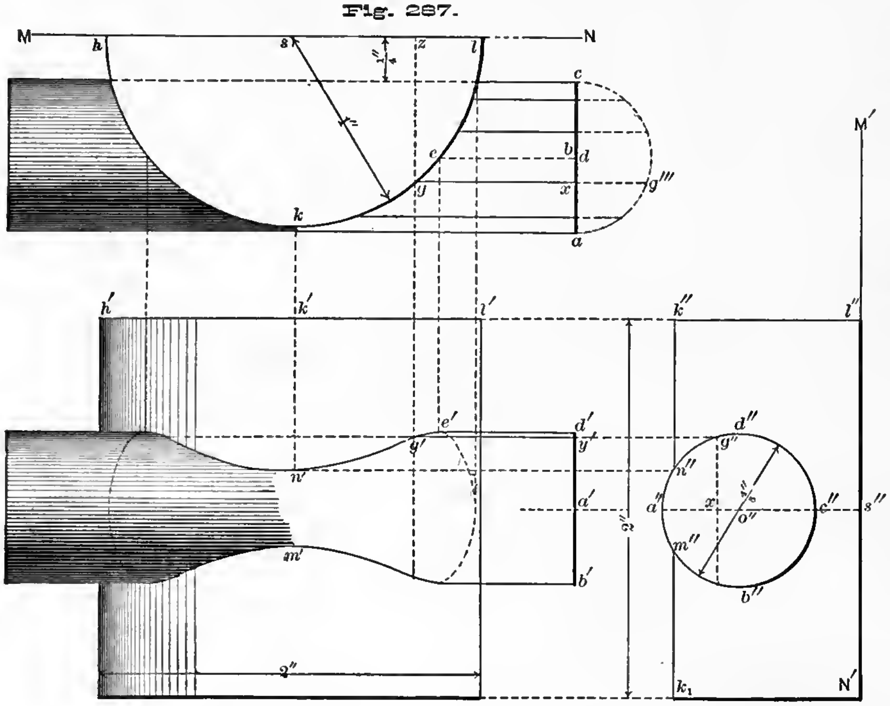
431. *To turn a right angle with a pipe by a four-piece elbow.* This problem would arise in carrying the blast pipe of a furnace around a bend. Except as to the number of pieces it differs but slightly from the last problem. Instead of *one* joint or curve of intersection there would be *three*, one less than the number of pieces in the pipe. (Fig. 286).

Let $o\,q\,s$ show the size of the cylinders employed, and be at the same time the plan of the vertical piece $o'\,s'\,n'\,a'$. Until we know where $a'\,n'$ will lie we have to draw $o'\,a'$ and $s'\,n'$ until they meet the elements from $S'$ and $T'$, and get the joint $m\,M'$ as for a two-piece elbow. On $m\,M'$ produced take some point $v'$, use it as a centre for an arc $t'\,x\,y\,t''$ tangent to the extreme elements; divide this arc, between the tangent points, into as many *equal* parts as there are to be joints in the turn; then tangents at $x$ and $y$—the intermediate points of division—will determine the outer limits of the joints at $a'$, $b'$ and $J$. Draw $a'\,v'$, finding $n'$ by its intersection with $s\,s'$; then $n'\,l'$ parallel to $a'\,b'$, and similarly for the next piece.

*The developments* of the smaller pieces would be equal, as also of the larger. One only is shown, laid out on the developed right section on $v'\,x$. The lettering makes the figure self-interpreting.

432. *The intersection of two cylinders, when each is partially exterior to the other.* The given condition makes it evident, by Art. 423, that a continuous non-plane curve will result.

Fig. 287.




Let one cylinder be vertical, 2″ in diameter and 2″ high. This is shown in half plan in $h\,k\,l$, and in front and side elevations between horizontals 2″ apart.

Let the second cylinder be horizontal; located midway between the upper and lower levels of the other cylinder; its diameter $\frac{4}{5}''$. On the side elevation draw a circle $a''\,b''\,c''\,d''$ of $\frac{4}{5}''$ diameter, locating its centre midway between $k''\,l''$ and $k_1\,N'$, and in such position that $a''$ shall be exterior to $k''\,k_1$. The elevation of the horizontal cylinder is then projected from its end view, and is shown in part without construction lines.

*The curve of intersection* is obtained by selecting particular elements of either cylinder and noting where they meet the other surface.

The foremost element of the vertical cylinder is $k \ldots k'\,n'\,m'$. Its side elevation, $k''\,k_1$, meets the circle at $n''$ and $m''$, which give the levels of $n'$ and $m'$ respectively.

On the horizontal cylinder the highest and lowest elements are central on the plan and meet the vertical cylinder at $e$, which projects down to the elements $d'$ and $b'$.

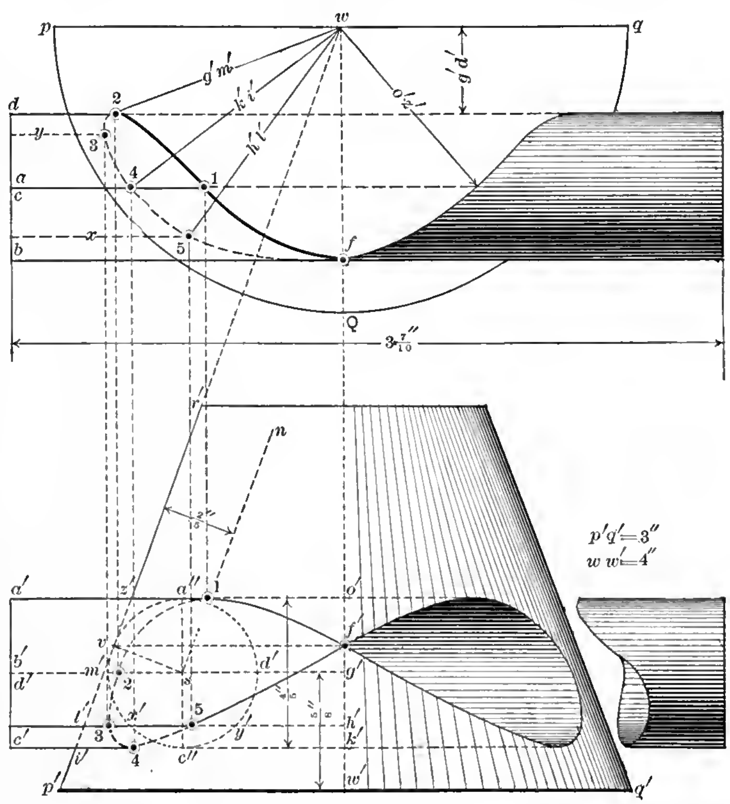
The front and rear elements, $c$ and $a$, would be central on the elevation. The vertical line drawn from the intersection of element $c$ with the arc $h\,k\,l$ gives the right-hand point of the curve of intersection, at the level of $a'$.

Any element as $g\,x$ may be taken at random, and its elevation found in either of the following ways: (a) Transfer $g\,z$, the distance of the element from $M\,N$, to $s''\,x$ on the side elevation, and draw $x\,g''$ and $g''\,y'$, to which last (prolonged) project $g$ at $g'$; or (b) prolong $g\,x$ to meet a

semi-circle on $a\,c$ at $g'''$; make $a'\,y' = x\,g'''$ and draw $y'\,g'$. The same ordinate $x\,g'''$, if laid off *below* $a$, would obviously give the other element which has the same plan $g\,x$, and to which $g$ projects to give another point of the desired curve.

433. *The intersection of a vertical cone by a horizontal prism.* Let the cone have an altitude, $w\,w'$, of 4″; diameter of base, 3″. (As the cylinder is entirely in front of the axis of the cone only one-half of the latter is represented.)

Fig. 288.




For the cylinder take a diameter of $\frac{4}{5}$″; length $3\frac{7}{10}$″; axis parallel to V, $\frac{5}{8}$″ above the base of the cone, and $\frac{2}{5}$″ from the foremost element. Draw $n\,s$ parallel to $p'\,r'$ and $\frac{2}{5}$″ from it; also $g'\,m'$ horizontal and $\frac{5}{8}$″ from the base; their intersection $s$ is the centre of the circle $a''\,d'\,c''\,m'$, of $\frac{4}{5}$″ diameter, which bears to the element $p'\,r'$ the relation assigned for the cylinder to the foremost element; said circle and $p'\,w\,w'$ are thus, practically, a *side elevation* of cylinder and cone, superposed upon the ordinary view.

The dimensions chosen were purposely such as to make one element of the cone tangent to the cylinder, that the curve of intersection might cross itself and give a mathematical "double point."

The width $d\,b$, of the plan of the cylinder, equals $m'\,d'$. The plan of the axis (as also of the highest and lowest elements, $a'$ and $c'$) will be at a distance $s\,g'$ from $w$. Any element as $x'\,y'\,h'$ is shown in plan parallel to $p\,q$, and at a distance from it equal either to $h'\,y'$ if on the rear or to $h'\,x'$ if on the front.

The element through $v$, on which $f'$ falls, is not drawn separately from $b\,f$ in plan, since $v\,f'$ and $m'\,g'$ are so nearly equal to each other; but $f$ must not be considered as on the foremost element of the *cylinder*, although it is apparently so in the plan.

*For the intersection* pass auxiliary horizontal planes through both surfaces; each will cut from the cone a circle whose intersection with cylinder-elements in the same plane will give points sought.

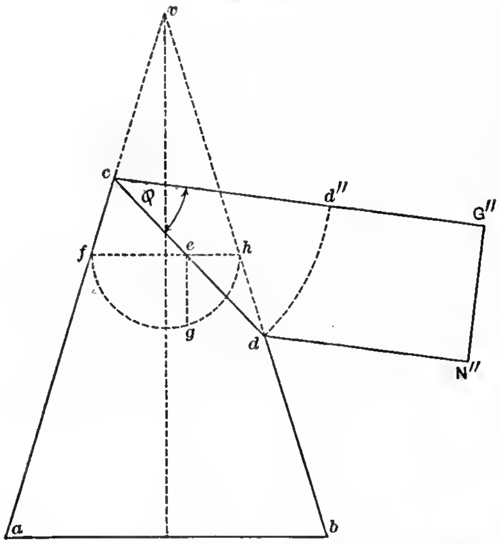
A horizontal plane through the element $a'$ would be represented by $a'\,o'$, and would cut a circle of radius $o'\,z'$ from the cone. In plan such circle would cut the element $a$ at point 1, and also at a point (not numbered) symmetrical to it with respect to $w\,Q$. Similarly, the horizontal plane through the element $x'\,h'$ cuts a circle of radius $l'\,h'$ from the cone; in plan such circle would meet the elements $x$ and $y$ in two more points (5 and 3) of the curve.

As the curve is symmetrical with respect to $w\,Q\,w'$ the construction lines are given for one-half only, leaving the other to illustrate shaded effects. The small shaded portion of the elevation of the cylinder is not limited by the curve along which it would meet the cone, but by a random curve which just clears it of the right-hand element of the cone.

434. *To find the diameter and inclination of a cylindrical pipe that will make an elbow with a conical*

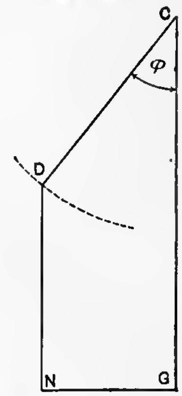
*pipe on a given plane section of the latter.* Let $vab$ be a vertical cone, and $cd$ the elliptical plane section on which the cylindrical piece is to fit. The diameter of the desired cylinder will equal the shorter diameter of the ellipse $cd$. To find this bisect $cd$ at $e$; draw $fh$ horizontally through $e$, and on it as a diameter draw the semi-circumference $fgh$; the ordinate $eg$ is the half width of the cone, measured on a perpendicular to the paper at $e$, and is therefore the radius of the desired cylinder.

Fig. 289.




In Fig. 290, the base $NG$ equals twice $ge$ of Fig. 289. At first indefinite perpendiculars are erected at $N$ and $G$, on one of which a point $C$ is taken as a centre for an arc of radius equal to $cd$ in Fig 289. The angle $\phi$ being thus determined is next laid off in Fig. 289 at $c$, and $cdN''G''$ made the exact duplicate of $CDNG$, completing the solution.

Fig. 290.



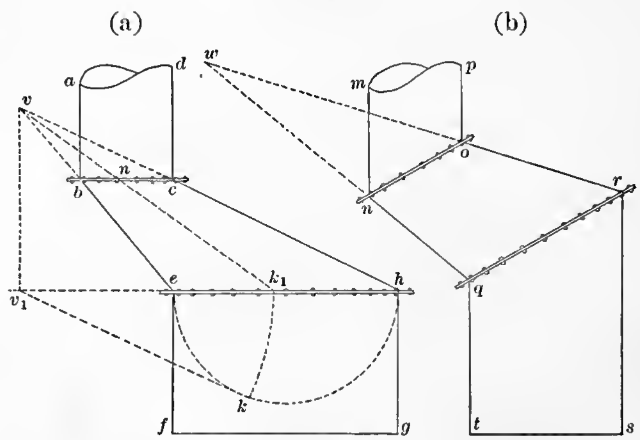

The developments are obtained as in Arts. 120 and 191.

435. *To determine the conical piece which will properly connect two unequal cylinders of circular section, whose axes are parallel, meeting them either* (a) *in circles or* (b) *in ellipses; the planes of the joints being parallel.*

(a) *When the joints are circles.* To determine the conical frustum $behc$ prolong the elements $eb$ and $hc$ to $v$; develop the cone $v\ldots eh$ as in Art. 418, and on each element as seen in the development lay off the real distance from $v$ to the upper base $bc$. Thus the element whose plan is $v_1k$ is of actual length $vk_1$ and cuts the upper base at a distance $vn$ from the vertex, which distance is therefore laid on $vk_1$ wherever the latter appears on the development.

(b) *When the joints are ellipses.* Let the elliptical joints $no$ and $qr$ be the bases of the conical piece $qnor$. To get the development complete the cone by prolonging $qn$ and $or$ to $w$; prolong $qr$ and drop a perpendicular to it from $w$; find the minor axis of the ellipse $qr$ as in the first part of Art. 434 and having constructed the ellipse proceed as in Art. 418, since in Fig. 255 the arc $abc\ldots g$ is merely a special case of an ellipse.

Fig. 291.

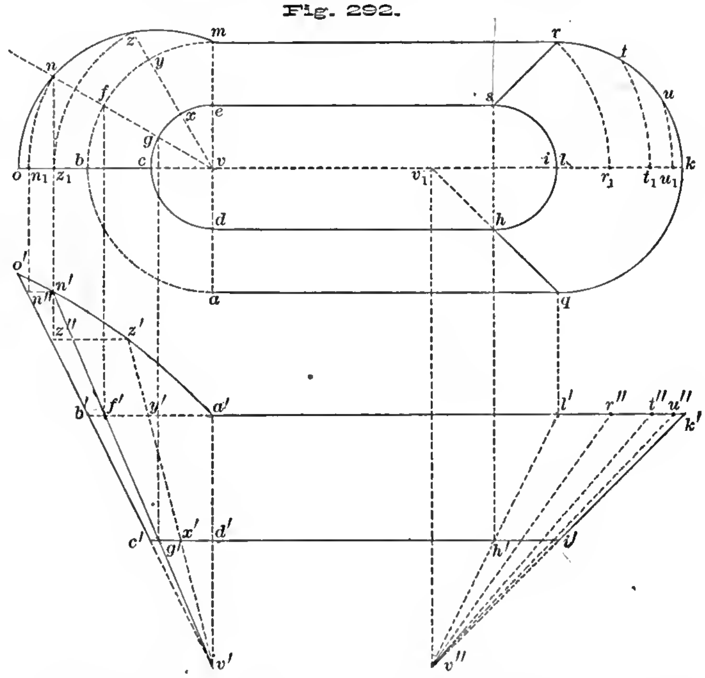



436. *The projections and patterns of a bath-tub.* Before taking up more difficult problems in the intersection of curved surfaces one of the most ordinary applications of Graphics is introduced, partly by way of illustrating the fact that the engineer and architect enjoy no monopoly of practical projections.

In Fig. 292 the height of the main portion of the tub is shown at $a'd'$. Let it be required that the head end of the tub be a portion of a *vertical right cone* whose base angle $c'b'a'$ equals the flare of the sides, such cone to terminate on a curve whose vertical projection is $o'n'z'a'$. Draw

two lines, $b'l'$ and $c'i'$, at first indefinite in length and at a distance $a'd'$ apart. Take $a'd'$ vertical, and regard it not only as the projection of the elements of tangency of the flat sides with the conical end, but also as the elevation of part of the axis, prolonging it to represent the latter. Use $v$, the plan of the axis, as the centre for a semicircle of radius $vc$, whose diameter $ed$ is the width of the bottom of the tub. Project $c$ to $c'$; make angle $v'c'd'$ equal to the predetermined flare of the sides; prolong $v'c'$ to $b'$ and $o'$; project $b'$ to $b$ on $vc$ prolonged and draw arc $abm$ with radius $bv$, obtaining $am$ for the width of the plan of the top.

Fig. 292.

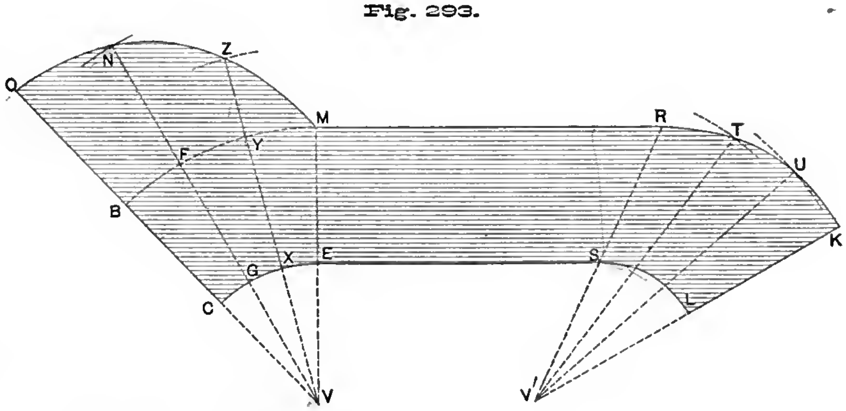


The plan of one-half the curve $o'n'z'a'$ is shown at $onzm$ and is thus found: Assume any element $v'x'y'$; prolong it to $z'$; obtain the plan $vxy$ and project $z'$ upon it at $z$. Similarly for $n$ and as many intermediate points as it might seem desirable to obtain.

Assuming that the foot of the tub is composed of an *oblique* cone whose section, $his$, with the bottom is equal to $ecd$, and whose base angle is $h'i'k'$, we project $i$ to $i'$, draw $i'k'$ at the given angle to the base, project $k'$ to $k$, and through the latter draw the semicircle $rkq$ with radius $bv$, obtaining the plan of the upper base.

Joining the tangent points $r$ and $s$, $h$ and $q$, we have $rs$ and $hq$ as the elements of tangency of sides with end. Their elevations coincide in $h'l'$, which meets $k'i'$ at $v''$, whose plan is $v_1$ on $hq$.

Fig. 293.



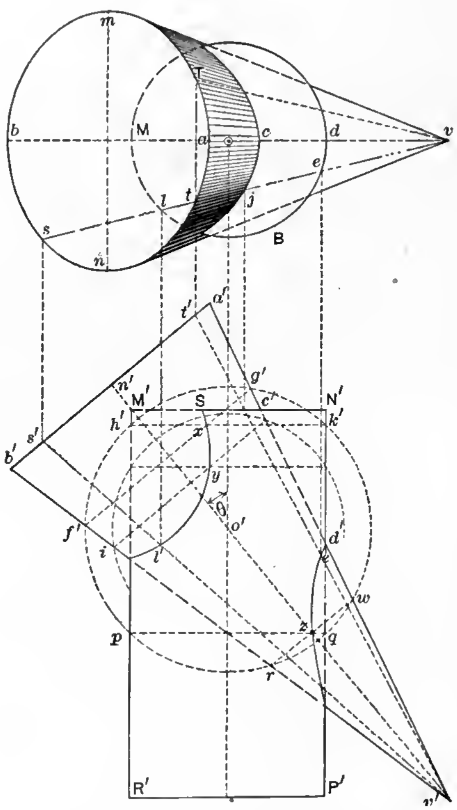

*The development.* Fig. 293 is the development of one-half of the tub. $EM$ equals $b'c'$; $VO$ equals $v'o'$; $VZ$ equals $v'z''$, the true length of $v'z'$, obtained, as in previous constructions, by carrying $z$ to $z_1$, thence to level of $z'$. Similarly at the other end. (Reference Articles 191, 408, 418.)

437. *The intersection of a vertical cylinder and an oblique cone, their axes intersecting.*

Let $MBd$ and $M'R'P'N'$ be the projections of the cylinder; $v'.a'b'$ and $v.anbm$ those of the cone. The axes meet $o'$ at an angle $\theta$ which is arbitrary.

The ellipse $a\,n\,b\,m$ is supposed to be constructed by one of the various methods employed when the axes are known; and in this case we get the *length* of $m\,n$ from $a'\,b'$ and its *position* from $n'$, while $a\,b$ is vertically above $a'\,b'$.

Fig. 294.



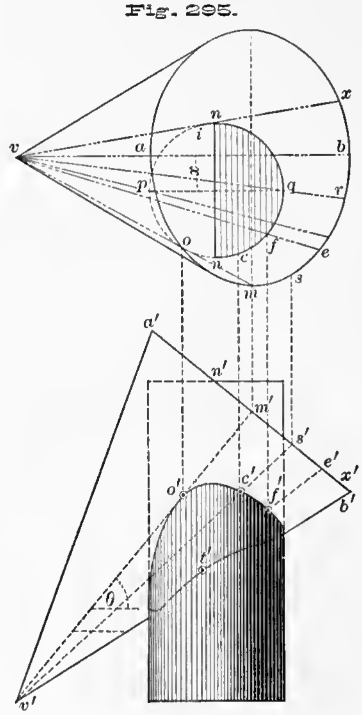

(a) *Solution by auxiliary vertical planes.* Any vertical plane $v\,l\,s$ will cut elements from the cylinder at $e$ and $l$; also, from the cone, elements which meet the base at $s$ and $t$. Project $s$ and $t$ to $s'$ and $t'$, join the latter with the vertex $v'$ and note $l'$ and $e'$ (just below $d'$) where they cross the vertical projection of the elements from $l$ and $e$; these will be points in the desired curve of intersection.

By assuming a sufficient number of vertical planes through $v$ the entire curve can be determined.

(b) *Solution by auxiliary spheres.* If two surfaces *of revolution* have a common axis they will intersect each other in a circle whose plane is perpendicular to that axis.* This property can be advantageously applied in problems of intersection.

With $o'$—the intersection of the axes—as a centre, we may draw circles with random radii $o'f'$, $o'i$, and let these represent spheres. The sphere $f'g'w$ intersects the *cone* in the circle $f'g'$; the *cylinder* in the circle $h'k'$. These circles intersect each other at $x$ in a common chord whose extremities are points of the curves sought. They are both projected in the point $x$.

Fig. 295.




A second pair of circular sections, lying on the same auxiliary sphere, are seen at $p\,q$ and $r\,w$, their intersection $z$ being another point in the solution. The point $y$ results from taking the smaller sphere.

438. *Intersection of a cylinder and cone, their axes not lying in the same plane.*

In Fig. 295 let the cylinder be vertical and the cone oblique, the axis of the latter being parallel to V and inclined $\theta°$ to H, and also lying at a distance $x$ back of the axis of the cylinder.

The auxiliary surfaces employed may preferably be vertical planes through the vertex of the cone, since each will then cut elements from both cylinder and cone. Thus, $v\,f\,e$ is the h. t. of a vertical plane which cuts $e\,v$, $e'v'$ from the cone, and the vertical element through $f$ from the cylinder; these meet in vertical projection at $f'$, one point of the desired curve. The *plan* of the intersection obviously coincides with that of the cylinder.

---

* By the definition of a surface of revolution (Art. 340) any point on it can generate a *circle* about its axis. If, then, two surfaces have *the same axis*, any point common to both surfaces would generate one and the same circle, which must also lie on both surfaces and therefore be their line of intersection.

439. *Conical elbow; right cones meeting at a given angle and having an elliptical joint.* This is one of the cases mentioned in Art. 423 as not admitting of illustration in the same way as when dealing with surfaces of uniform cross section, but a plane intersection is nevertheless secured as with cylinders by making the *extreme elements* of the cones intersect.

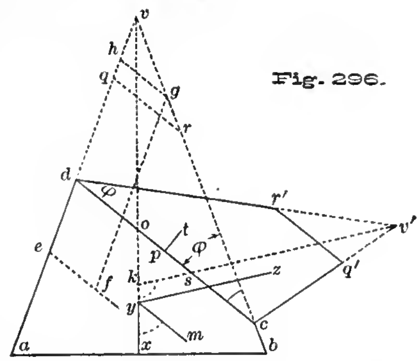


Fig. 296.

Let $vx$ in Fig. 296 be the axis of one of the cones. If $xyz$ is the required angle between the axes bisect it by the line $ym$, and draw the joint $cd$ parallel to such bisector. The right cone which is to meet $abcd$ on $cd$ must be capable of being cut in a section equal to $cd$ by a plane making an angle $\theta$ with its axis, and must obviously have the same base angle as the original cone; since, however, the upper portion $vdc$ of the given cone fulfills these conditions we may employ it instead of a new cone, rotating it about an axis $pt$ which is perpendicular to the plane of the ellipse $dc$ and passes through its centre. The point $o$, in which the axis $vx$ meets the plane $dc$, will then appear at $s$, by making $op = ps$; $sr'$, drawn parallel to $yz$, will be the new direction of $vo$; and an arc from centre $d$ with radius $cv$ will give $v'$, which is then joined with $d$ and $c$ to complete the construction.

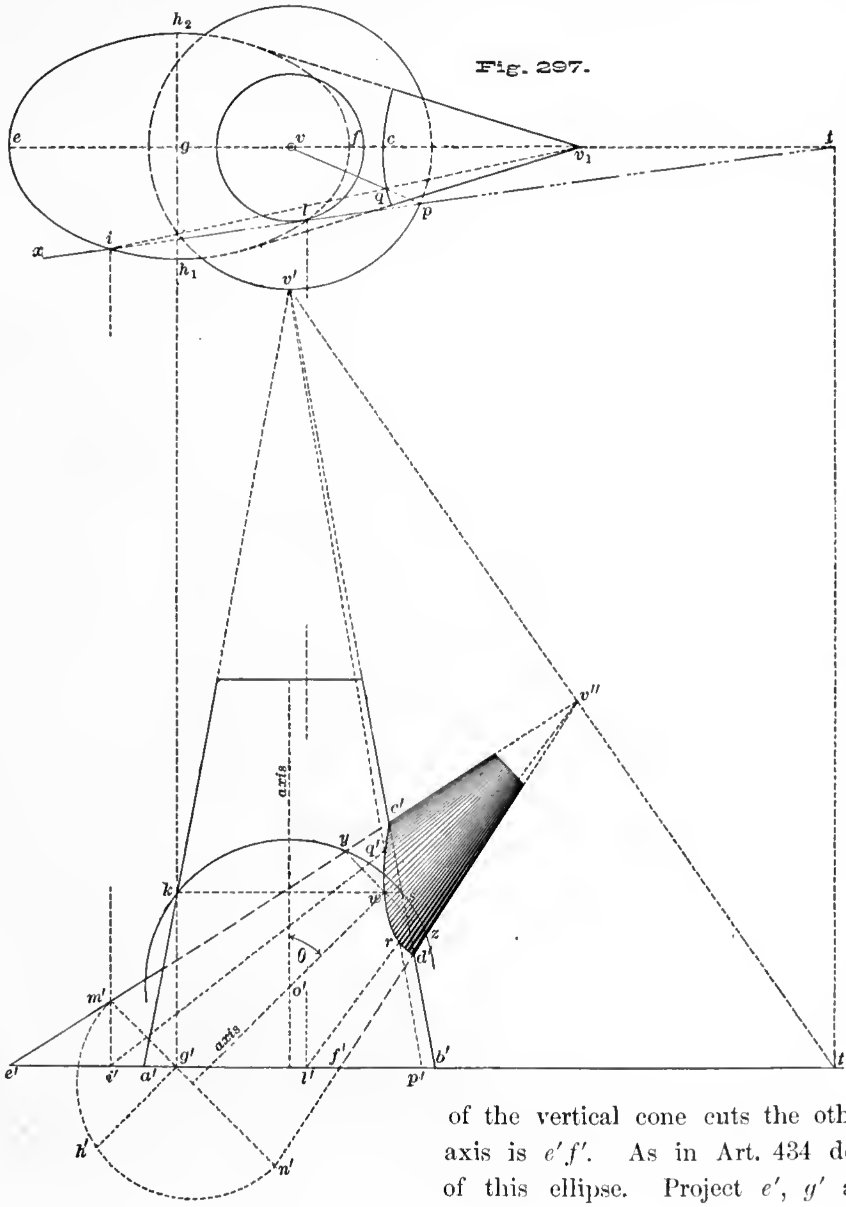


Fig. 297.

If the length of the major axis of the elliptical joint had been assigned, as $ef$ for example, that length would have first been laid off from some point $e$ on the extreme element and parallel to $ym$, then from $f$ a parallel to $ve$, giving $g$ on $vc$; then $gh$ parallel and equal to $ef$, gives the joint in its proper place.

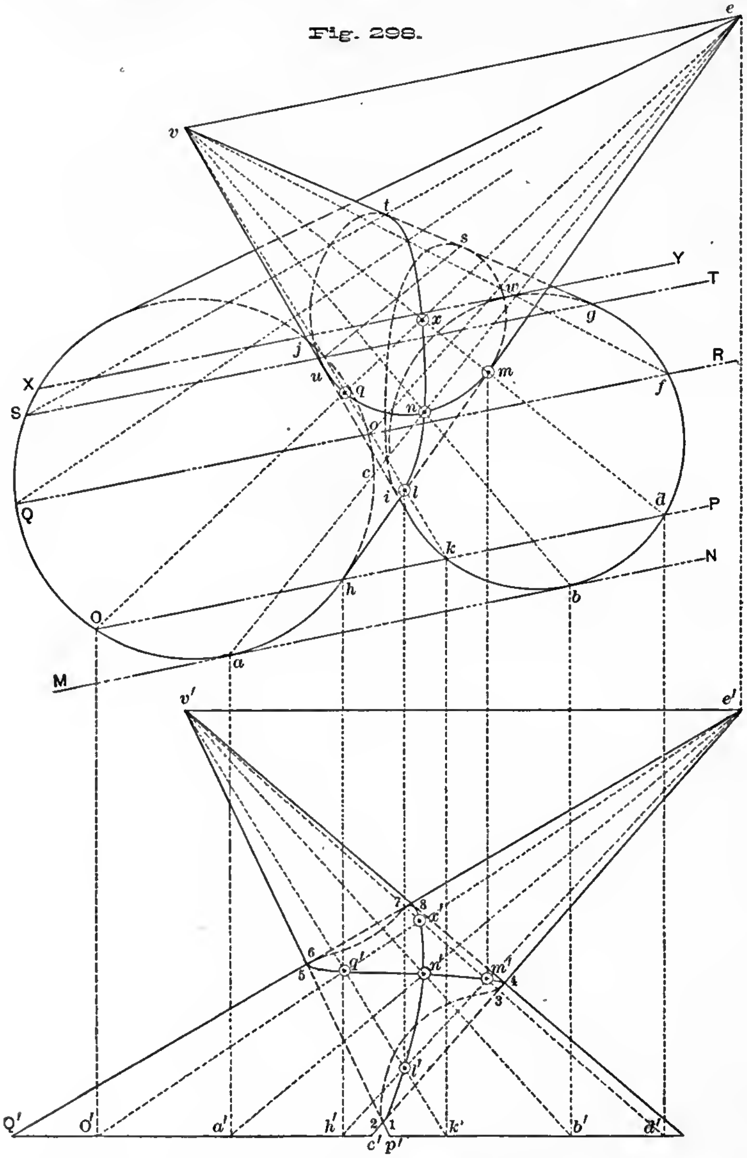
440. *Right cones intersecting in a non-plane curve; axes meeting at an oblique angle.* Let one cone, $v'.a'b'$, (Fig. 297) be vertical; the other, oblique, its axis meeting $v'o'$ at an angle $\theta$.

The plane $a'b'$ of the base of the vertical cone cuts the other cone in an ellipse whose longer axis is $e'f'$. As in Art. 434 determine $g'h'$, the semi-minor axis of this ellipse. Project $e'$, $g'$ and $f'$ up to $e$, $g$ and $f$; make

$g h_1$ and $g h_2$ each equal to $g'h'$; then on $ef$ and $h_1 h_2$ as axes construct the ellipse $e\,h_1 f\,h_2$ as in Art. 131. Tangents from $v_1$ to the ellipse complete the plan of the oblique cone.

(a) *The curve of intersection, found by auxiliary planes.* In order that each auxiliary plane shall contain an element (or elements) of each cone, it must contain both vertices and therefore the line $v'v''$, which joins them; hence its trace on the plane $e'a'b'$ must pass through the trace, $t't$, of such line on that plane. Take $tx$ as the horizontal trace of one of these auxiliary planes. It cuts elements starting at $i$ and $l$ on the base of the oblique cone. One of the elements cut from the other cone is $v\,p$, which in vertical projection ($v'\,p'$) crosses the elevations of the other elements at $q'$ and $r'$, two points of the curves sought. Since the extreme elements of the cones are parallel to V we will have $c'$ and $d'$—the intersections of their elevations—for two more points of the curve. Having found other points by repeating the same process the curve $c'\,q'\,r\,d'$ is drawn through them, and the cones may then be developed as in Art. 191.

Fig. 298.




(b) *Method by auxiliary spheres.* Since the axes intersect we may use auxiliary spheres as in Case (b) of Art. 437. Thus, with $o'$—the intersection of the axes—as a centre, take any radius $o'k$ and regard arc $k\,y\,z$ as representing a portion of a sphere which cuts the cones in $k\,s$ and $y\,z$. These meet at $w$, one point of the curve of intersection $c'\,q'\,d'$.

441. *Intersecting cones, bases in the same plane but axes not.* Let $v.\,k\,b\,f\,g$ and $c.\,s\,Q\,h\,j$ be the plans of the cones; $v.'\,p'\,d'$ and $e.'\,Q'\,c'$ their elevations.

As argued in Case (a) of the last problem, the auxiliary planes must contain the line joining the vertices; their H-traces would therefore, in the general case, pass through the trace of that line upon the plane of the bases; but, in the figure, both vertices having been taken at the same height above the bases, the line which joins them must be *horizontal*, hence *parallel* to the H-traces of the auxiliaries: that is, $X\,Y$, $S\,T$, $Q\,R$, etc., are *parallel* to $v\,e$.

It happens that the trace $M\,N$ of the foremost auxiliary plane is *tangent* to both bases, hence contains but one element of each cone and determines but one point of the desired curve. These elements, $a\,e$ and $b\,v$, meet at $n$, while their elevations intersect at $n'$.

Each of the other planes, except $XY$, being secant to both bases, will cut two elements from each cone, their mutual intersections giving four points of the curve of interpenetration. Thus, in plane $OP$, the element $Oe$ meets $vk$ in $q$ and $vd$ in $x$, while element $he$ gives $l$ and $m$ on the same elements.

The plane $XY$ being tangent to one base while secant to the other gives but two points on the curve sought.

*Order of connecting the points.* Starting with any plane, as $MN$, we may trace around the bases either to the right or left. Choosing the former we find, in the next plane, the point $h$ to the right of $a$ on one base, and $d$ similarly situated with respect to $b$ on the other; therefore $m$, on $he$ and $dv$, is the next point to connect with $n$. Elements $oe$ and $fv$ give the next point, then $ue$ and $gv$ locate $s$, after which those from $j$ and $w$ give the last before a return movement on the base of the $v$-cone. As nothing new would result from retracing the arc $gfd$ we continue to the left from $w$, although compelled to retrace on the other base, since planes beyond $j$ would not cut the $v$-cone. The element $ue$ is therefore taken again, and its intersection noted with an element whose projection happens to be so nearly coincident with $vx$ that the latter is used.

Continuing along arcs $och$ and $ikb$ we reach the plane $MN$ again, the curves $ilx$ and $qnm$ crossing each other then at $n$—the point lying in that plane. Such point is called a *double point*, and occurs on non-plane curves of intersection at whatever point of two intersecting surfaces they are found to have a common tangent plane.

Tracing to the left from $a$ and to the right from $b$ the elements $Oe$ and $dv$ are reached, in the plane $OP$. Their intersection $x$ is joined with $n$ on one side and with the intersection of $Se$ and $gv$ on the other. Soon the tangent plane $XY$ is again reached and a return movement necessitated, during which the arc $XSQOa$ is retraced, while on the other base the counter-clockwise motion is continued to the initial point $b$, completing the curve.

*Visibility.* The visible part of the intersection in either view must obviously be the intersection of those portions of the surfaces which would be visible were they separate, but similarly situated with respect to H and V.

In plan the point $n$ lies on visible elements, and either arc passing through it is then visible till it passes (becomes tangent to, in projection) an element of extreme contour as at $m$ or $t$, when it runs from the upper to the under side of the surface and is concealed from view.

The point $w$ would be visible on the $v$-cone but for the fact that it is on the under side of the $e$-cone.

A similar method of inspection will determine the visible portions of the vertical projection of the curve, which will not be identical with those of the plan. In fact, a curve wholly visible in one view might be entirely concealed in the other.

442. *The intersection of a vertical cylinder and an oblique cone, their axes in the same plane.* If in Art. 440 the vertex $v'$ were removed to infinity the $v$-cone would become a vertical cylinder; the line $v'v''$ would become a vertical line through $v''$; $t$ would be vertically above $v''$; but the *method* of solving would be unchanged.

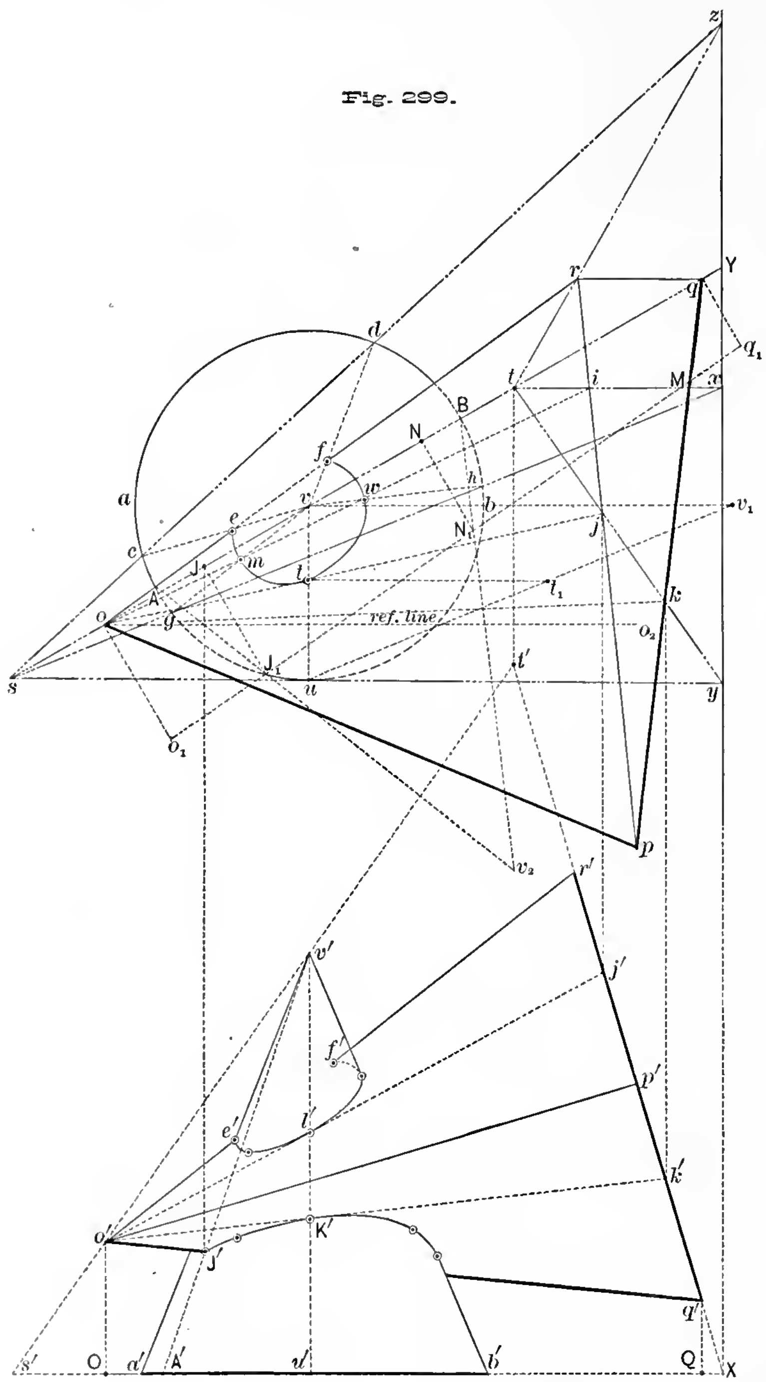
443. In general, any method of solving a problem relating to a cone will apply with equal facility to a cylinder, since one is but a special case of the other. The line, so frequently used, that passes through the vertex of a cone in the one problem is, in the other, a parallel to the axis of the cylinder. Planes containing both vertices of cones become planes parallel to both axes of cylinders.

In view of the interchangeability of these surfaces it is unnecessary to illustrate by a separate figure all the possible variations of problems relating to them.

444. *Intersection of two cones, two pyramids, or of a cone and a pyramid, when neither the bases nor axes lie in one plane.*

One method of solving this problem has been illustrated in Art. 429, where the intersection was found by using auxiliary planes that were either vertical or perpendicular to V; we may as easily, however, employ the method of the last problem, viz., by taking auxiliary planes so as to contain both vertices. This will be illustrated for the problems announced, by taking a cone and pyramid; and, for convenience, we will locate the surfaces so that one of them will be vertical, and the base of the other will be perpendicular to V, since the problem can always be reduced to this form.

Fig. 299.




Let the cone $v'.a'b'$, $v.cdB$, (Fig. 299) be vertical, and the pyramid $o'.r'q'p'$, $o.rqp$, inclined.

We will assume that the projections of the pyramid have been found as in preceding problems, from assigned data, using $oo_2$, $o'p'$, (taken perpendicular to the base $r'q'$) as the reference line.

Join the vertices by the line $v'o'$, $vo$, and prolong it to get its traces, $ss'$ and $tt'$, upon the planes of the bases. All auxiliary planes containing the line $vo$, $v'o'$, must intersect the planes of the two bases in lines passing through such traces.

Prolong $r'q'$ to meet the plane $a'b'$ at $X$. Project up from $X$, getting $yz$ for the plan of the intersection of the two bases.

We may assume any number of auxiliary planes, some at random, but others more definitely, as those through edges of the pyramid or tangent to the cone. Taking first one through an edge, as $or$, we have $trz$ for its trace on the pyramid's base, then $zs$ for its trace on H. The elements $cv$ and $dv$ which lie in this plane meet the edge $or$ at $e$ and $f$, giving two points of the curve. These project to $o'r'$ at $e'$ and $f'$.

The plane $s\,y$, tangent to the cone along the element $u\,v$, has the trace $y\,t$ on the base of the pyramid, and cuts lines $j\,o$ and $k\,o$ from its faces. These meet $v\,u$ at two more points of the curve, their elevations being found by projecting $j$ to $j'$ and $k$ to $k'$, drawing $o'\,j'$ and $o'\,k'$, and noting their intersections with $v'\,u'$. To check the accuracy of this construction for either point, as $l$, draw $v\,v_1$ perpendicular to $v\,u$ and equal to $v'\,u'$, join $v_1$ with $u$, and we have in $v\,v_1\,u$ the rabatment of a half section of the cone, taken through the element $v\,u$ and the axis; then $l\,l_1$, parallel to $v\,v_1$, will be the height of $l'$ above the base $a'\,b'$.

With one exception, any auxiliary plane between $s\,y$ and $s\,z$ will give four points of the intersection. The exception is the plane $s\,Y$, containing the edge $o\,q$, and which, on account of happening to be vertical, requires the following special construction if the solution is made wholly on the plan: Rabat the plane into H; the elements it contains will then appear at $A\,v_2$ and $B\,v_2$, while the edge $o\,q$ will be seen in $o_1\,q_1$ (by making $o\,o_1 = o'\,O$, and $q\,q_1 = q'\,Q$); elements and edge then meet at $J_1$ and $N_1$ which counter-revolve to $J$ and $N$. We might, however, get elevations first, as $J'$, by the intersection of element $A'\,v'$ with edge $o'\,q'$; then $J$ from $J'$.

In the interest of clearness several lines are omitted, as of certain auxiliary planes, hidden portions of the ellipses, and the curves in which $s\,r\,q$ (the rear face) cuts the cone. The student should supply these when drawing to a larger scale.

*See the latter part of this chapter for further problems on the intersection of surfaces.*

## MONGE'S DESCRIPTIVE GEOMETRY.—FIRST ANGLE METHOD.

445. In this method—the first, and so long the only one employed, and whose use would probably be still universal but for the reason given in Art. 383—the object is located in front of the vertical plane and above the horizontal, as illustrated in Art. 385.

While acquaintance with what has preceded in this chapter would be an advantageous preliminary to the study of the First Angle treatment of figures, yet it is not absolutely essential; but if for any reason, as, for example, with reference to its applications to perspective or stone cutting, the First Angle Method is taken up in advance of the other, it is assumed that the student will first thoroughly familiarize himself with Arts. 284–330, and 335–379.

446. *To determine one projection of a point on a given surface, having given the other.* This problem is of frequent recurrence and is illustrated in Figs. 300 and 301, in which the more familiar surfaces are shown in their most elementary positions, i. e., with axes either perpendicular to or parallel to a plane of projection.

The *required* projection is in each case enveloped in a small circle.

Where two solutions are possible both are given. The general solution of this problem, for all surfaces, is as follows: Through the *given* projection draw a line on the surface, preferably a straight line, but otherwise the simplest curved section possible; obtain the other projection of this auxiliary line and project upon it from the given projection.

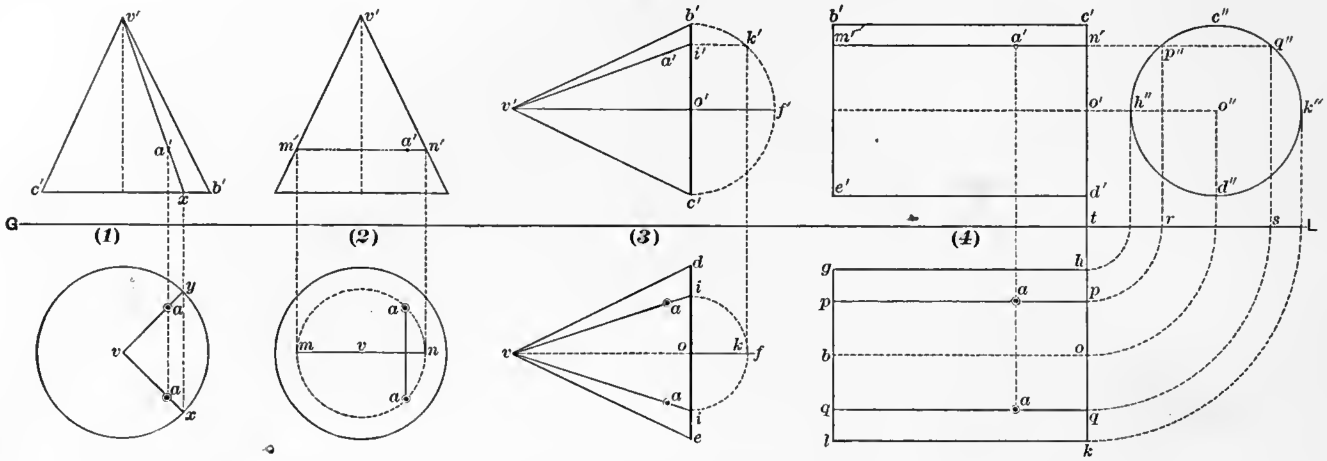
(a) *Right cone, axis vertical.* In No. 1 of Fig. 300 the element $v'\,x'$ is drawn through the given projection $a'$. Projecting $x'$ down to $x$ and $y$ we draw the plans $v\,x$ and $v\,y$ and project $a'$ upon each.

(b) *Right cone, axis vertical. Solution by auxiliary circle.* In No. 2 draw through the given projection $a'$ the line $m'\,n'$ parallel to the v. p. of the base. It represents a circle of diameter $m'\,n'$, which is seen in full size in plan, and upon which $a'$ projects in the two possible solutions.

(c) *Right cone, axis parallel to the ground line.* As before, $a'$ represents the given projection. The

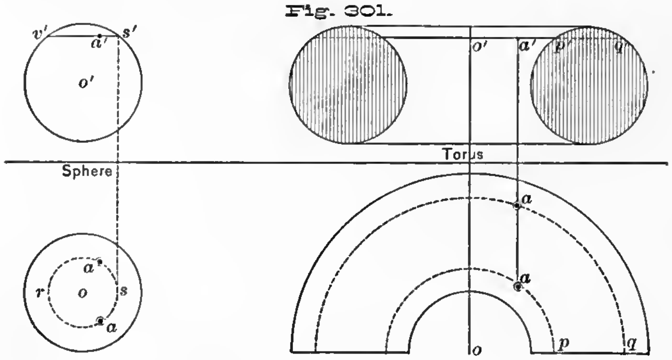
element $v'a'$ meets the base at $i'$, whose real distance in front of or to the rear of the vertical diameter of the base is seen at $i'k'$, found by rotating the semi-base on $b'c'$ as an axis until it is seen in full size at $b'f'c'$. The counter-revolution is shown in plan, and the two solutions indicated.

Fig. 300.




(d) *Right cylinder, axis parallel to the ground line.* In No. 4 the two rectangles represent the projections of the cylinder on H and V. To find the plans of the elements whose common elevation passes through $a'$ rotate the end of the cylinder into V, using as an axis the vertical line $tc'$. (The vertical-diameter method of the last case would answer equally well.) The arcs show the paths of the various points. Then $m'n'$ projects to both $p''$ and $q''$, which are transferred to $p$ and $q$ by arcs from $r$ and $s$.

Fig. 301.




(e) *Sphere.* In Fig. 301 a horizontal section through the given projection, $a'$, cuts a circle seen in full size in plan, upon which $a'$ projects in the two solutions.

(f) *Annular torus.* This surface, also known as the anchor ring, is generated by revolving a circle about an axis in its plane but not a diameter. It has the same mathematical properties whether the axis is exterior to the circle or is a tangent or a chord; but obviously there would be no hole in the surface except in the former case.

In Fig. 301 one-half of the ring is shown in plan and elevation, either shaded section showing the size of the generating circle. The axis is a vertical line through $o$.

The axis being vertical, a horizontal plane through $a'$ will cut two circles from the torus, of radii equal respectively to $o'p'$ and $o'q'$. These are seen full size in plan, and upon them $a'$ projects.

447. As the representation of any surface of revolution, when its axis is oblique to one or both planes of projection, necessitates the drawing of the oblique projection of a circle, the solution of the latter problem is a natural preliminary to constructions involving the former.

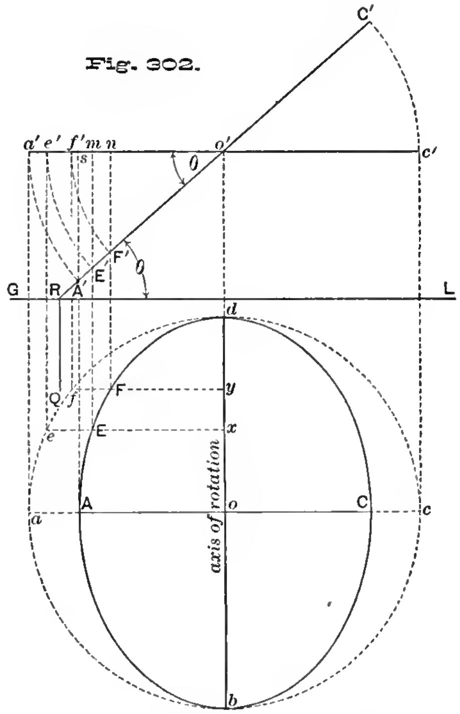
448. *To obtain the projection of a circle when its plane is oblique to the plane of projection. Proof that such projection is an ellipse.* In Fig. 302 let $abcd, \ldots a'c'$, be the projections of a circle lying in a horizontal plane. Using as an axis of rotation the diameter $bd$, which is perpendicular to V, let

us suppose the plane of the circle to rotate through an angle $\theta$; $QRC'$ will then represent its new position.

Since the axis is perpendicular to V any points, as $a$ and $c$, of the original circle, will describe arcs parallel to V and therefore seen in their true size in elevation, as at $a'A'$ and $e'E'$, while their plans, $aA$ and $eE$, will be parallel to $G.L$; their new positions, $A$, $E$, are then, evidently, the intersections of verticals from $A'$ and $E'$ with horizontals through $a$ and $c$.

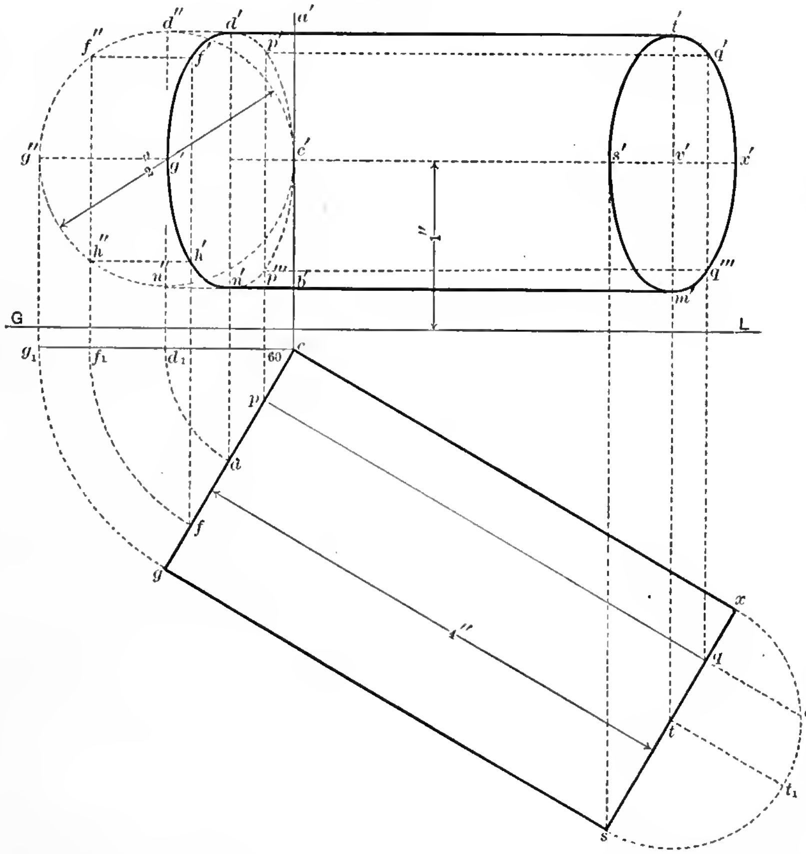
In Analytical Geometry the "greater auxiliary circle" of an ellipse has for its diameter the major axis of the latter curve; and, by analysis, the relation is established that, when measured on the same perpendicular to such major axis, an ordinate of the circle will be to the corresponding ordinate of the ellipse as the major axis of the ellipse to its minor axis. If, therefore, we can establish this relation between the circle $abcd$ and the curve $AbCd$, the latter must be an ellipse.

Fig. 302.




In the elevation we have, from similar triangles, the proportion $o'E' : o'm :: o'A' : o's$. But $o'E' = o'c' = ex$; $o'm = Ex$; $o'A' = o'a' = oa$; and $o's = oA$ : the proportion may therefore be written $ex : Ex :: oa : oA :: 2oa (= bd) : 2oA (= AC)$.

Fig. 303.



449. *Working drawing of a horizontal cylinder 4″ long, 2″ in diameter, axis 1″ above* H *and inclined 30° to* V. Draw first the plan $cgsx$ (Fig. 303) which is simply a rectangle 2″ × 4″, with longer sides at 30° to the ground line. The ends $cg$ and $sx$ are circles 2″ in diameter and vertical. Rotate the base $cg$ about the vertical *tangent* $c.a'b'$ until it takes the position $cg_1$, when its elevation $g''n''c'd''$ will equal the circle of which it is the projection. The centre of such circle will be at the height (1″) assigned for the axis. Note various points, as $g_1g''$, $f_1f''$, $d_1d''$, and then, by a construction in strict analogy to that of the last problem, counter-revolve them into the original plane. Their paths of rotation will be horizontal arcs, seen in full size in plan, but as horizontal straight lines in elevation, as $g''g'$, $f''f'$, $d''d'$.

The new elevations are then the intersections of verticals from $g$, $f$, $d$ to the levels at which they rotated, giving points of the ellipse $g' f' d' e'$.

The other end of the cylinder might have been obtained similarly, but the figure illustrates the use of the horizontal diameter $s\,x$, $s'x'$, as an axis, when $q\,q_1$ shows the distance to lay off above and below $v'$ to get the levels of the elements whose common plan is $p\,q$; that is, for $p'q'$ and for $p''' q'''$.

450. *To project a circle when its plane is oblique to both planes of projection; also to draw a tangent at a given point.* To avoid multiplicity of lines we will assume that in Fig. 304 $P\,Q\,P'$—the plane of the circle—has been already determined from assigned inclinations, by means of Art. 319. Let it be required that the circle lying in that plane shall have a given radius ($o\,d$), and that its centre shall be at assigned distances, $b'b$ and $o\,n$, from H and V respectively. To fulfill the condition as to height of centre draw $a'b'$ at assigned height $b'b$ above G. L., to represent the v. p. of a horizontal of the plane. On the plan $a\,b$ of such horizontal note the point $o$ which fulfills the condition as to assigned distance ($o\,n$) from V; this will be the h. p. of the centre and projects to $o'$.

By Art. 306 rabat $o$ into $H$, about $P\,Q$ as an axis. It takes the position $o_1$, about which draw a circle with the prescribed radius $o\,d$.

The diameter $d_1 f_1$, which is parallel to the axis $P\,Q$, remains so during counter-revolution, and at $d\,f$ (passing through the original $o$) becomes the major axis of the ellipse, since it is the only diameter which is horizontal and therefore projected on H in its actual size. Project $d$ and $f$ to $d'$ and $f'$ on $a'b'$, since its elevation must evidently be parallel to G. L.

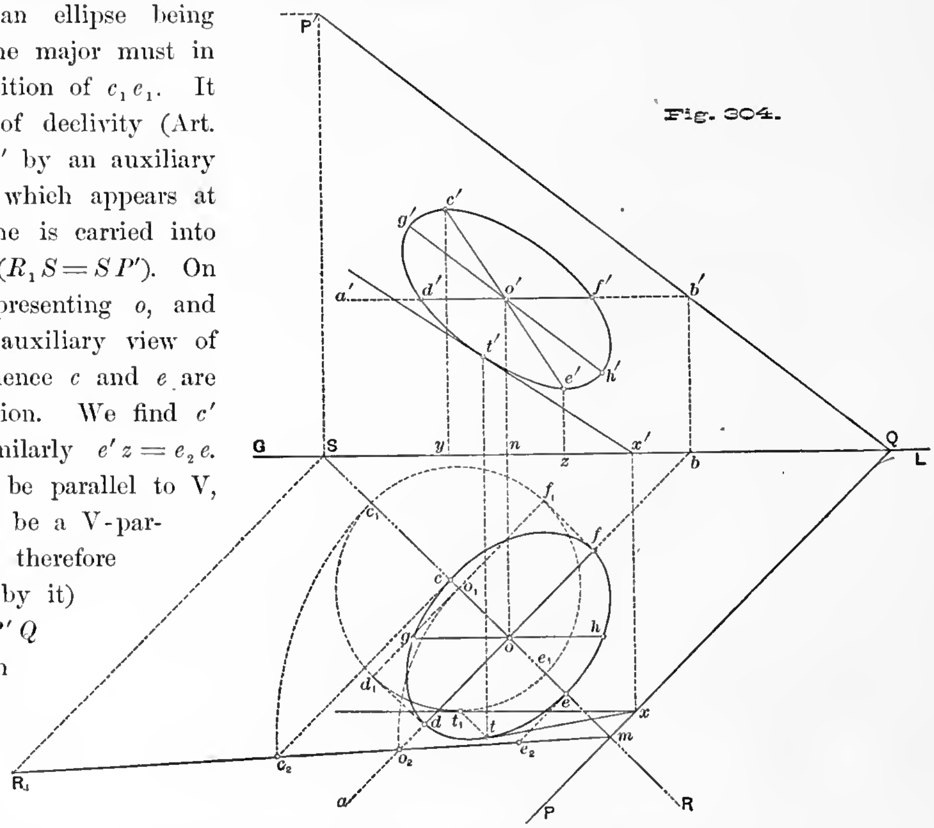


Fig. 304.

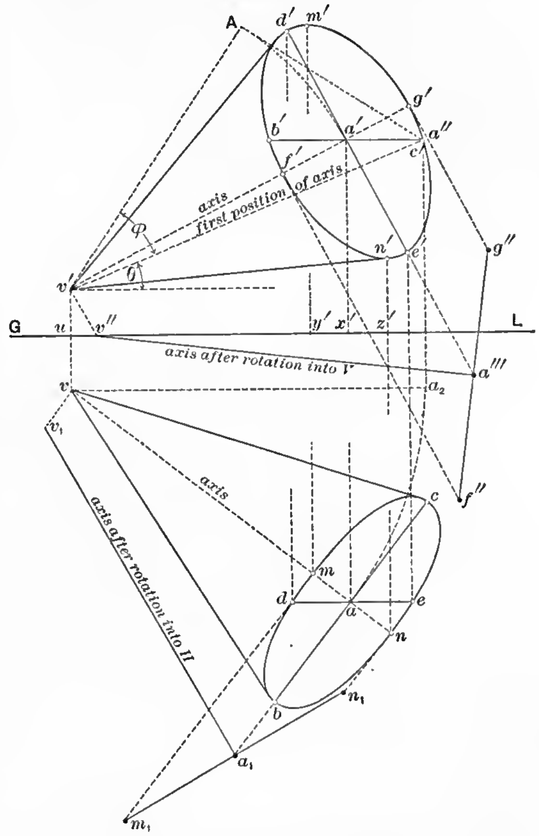
The minor axis of an ellipse being always perpendicular to the major must in this case be the space-position of $c_1 e_1$. It will be part of the line of declivity (Art. 301) cut from plane $P\,Q\,P'$ by an auxiliary *vertical* plane $R\,S\,P'$, and which appears at $m\,R_1$ when the latter plane is carried into H about $R\,S$ as an axis. ($R_1 S = S\,P'$). On such line we find $o_2$ representing $o$, and make $c_2 e_2 = d\,f$ for the auxiliary view of the minor axis sought, whence $c$ and $e$ are derived by counter-revolution. We find $c'$ at height $c'y = c_2 c$; similarly $e'z = e_2 e$.

But one diameter can be parallel to V, and in this case it must be a V-parallel (Art. 300) of $P'\,Q\,P$; therefore through $o'$ (and bisected by it) draw $g'h'$ parallel to $P'Q$ and of length $d\,f$. Its plan $g\,h$ is parallel to G. L.

*To draw a tangent* at any point, as $t$, find $t_1$ from which $t$ was derived, and draw $t_1 x$ tangent to the *circle*. As $x$ is on the axis $P\,Q$ it remains constant during rotation of the plane, and, when $t_1$ returns to $t$, the tangent becomes $t\,x$, whence $t'x'$ as usual.

451. *To project a cone whose axis is inclined* $\theta°$ *to* H *and* $\phi°$ *to* V; *altitude* 3″; *diameter of base,* 2″. By Art. 309 we find $v\,a$ and $v'a'$, (Fig. 305), the projections of the axis. Although all diameters of the base are perpendicular to the axis of the cone only one can be so projected in each view, but it will be that one which, being parallel to the plane of projection, is seen in actual size (Art. 311); therefore make $b\,c$ and $d'e'$ each 2″ long, and perpendicular respectively to $v\,a$ and $v'a'$. Their other projections are parallel to G. L.

Fig. 305.




The shorter axis of the base, in plan, lies in a *vertical* meridian plane (Art. 340); in the elevation it is in the meridian plane perpendicular to V. Using the former for illustration, rabat it into H, when $v\,a$ will appear at $v_1 a_1$, by making $v\,v_1 = v'u$, and $a\,a_1 = a'x'$. Then $m'n'$, 2″ long and perpendicular to $v_1 a_1$, is the auxiliary view of the diameter which, in counter-revolution, appears as the shorter axis $m\,n$ of the ellipse. Make $y'm' = m\,m_1$, and $z'n' = n\,n_1$ to get $m'$ and $n'$.

From $f''g''$ we get $f'g'$, the minor axis of the elevation of the base, by a construction in strict analogy with that described for $m\,n$.

The determination of other points of each view is after the method of Art. 450.

452. *To determine an intermediate or guide pulley to connect a pair of band-wheels running on two lines of shafting which make any angle with each other.*

This machine-shop problem is introduced here to show one of the practical applications of Art. 450. The principle of the construction is based both on theory and experience, and is, briefly stated, that the belt *must be led on* to a wheel *in its plane.* That is, the *point* where the belt *leaves* the mid-plane of one wheel must lie *in the mid-plane* of the next wheel. If rotation is to be reversible the mid-plane of the intermediate pulley will be determined by tangents to each of the main pulleys from some point of the line in which their mid-planes intersect.

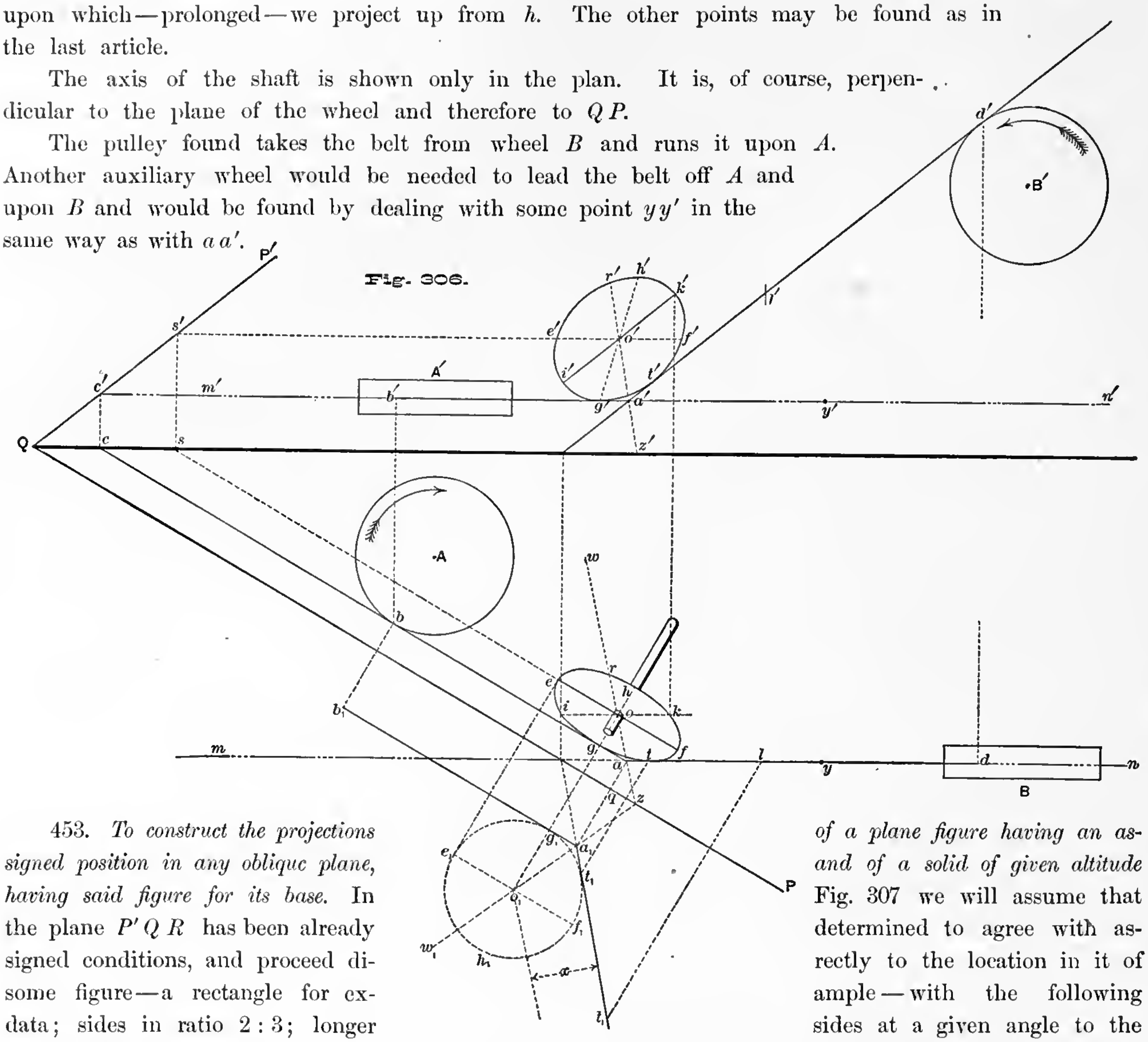
In Fig. 306 let A and B be the given pulleys, rotating on vertical and horizontal axes respectively, and in the directions indicated by the arrows. Their mid-planes intersect in the line $m\,n$, $m'n'$, at *any* point of which, as $a\,a'$, it is possible to draw tangents to both wheels; two such tangents, as $a\,b$, $a'b'$, and $a\,d$, $a'd'$, will therefore determine the mid-plane of an intermediate pulley which will direct the belt as desired.

Determine $P'Q\,P$, the plane of these tangents, and rabat it about $Q\,P$ into H; $a\,b$ then becomes $a_1 b_1$, and $a\,d$ appears (partially) at $a_1 l_1$ (Art. 306). Bisect the angle $b_1 a_1 l_1$ by the line $w_1 z$, and on it find a point $o_1$ with which as a centre a circle can be drawn that will be tangent to both $a_1 b_1$ and $a_1 l_1$, its radius $x$ to be that assigned for the pulley. After counter-revolution the circle appears as the ellipse $e\,f\,g\,h$, found by the method of the last article, with, however, the following special features which somewhat simplify the solution: First, the bisector $w_1 z$ becomes $z\,w$, $z$ being on the axis; hence $o_1$ projects directly to $z\,w$ at $o$; second, as tangent $a\,b$ is parallel to $Q\,P$ we have in $o\,g$ the semi-minor axis, and avoid a separate construction to obtain it.

For the elevation project $z$ to $z'$, draw $z'a'$ and project $o$ upon it. Next $g$ to $g'$ and draw $g'o'$, upon which—prolonged—we project up from $h$. The other points may be found as in the last article.

The axis of the shaft is shown only in the plan. It is, of course, perpendicular to the plane of the wheel and therefore to $QP$.

The pulley found takes the belt from wheel $B$ and runs it upon $A$. Another auxiliary wheel would be needed to lead the belt off $A$ and upon $B$ and would be found by dealing with some point $yy'$ in the same way as with $aa'$.




453. *To construct the projections of a plane figure having an assigned position in any oblique plane, and of a solid of given altitude having said figure for its base.* In Fig. 307 we will assume that the plane $P'QR$ has been already determined to agree with assigned conditions, and proceed directly to the location in it of some figure—a rectangle for example—with the following data; sides in ratio $2:3$; longer sides at a given angle to the horizontal; lowest corner at a given height $tt'$ from H.

Let $M'NM$ be an auxiliary vertical plane, cutting $P'QR$ in a line of declivity which appears at $ns_1$ after rabatment on $MN$.

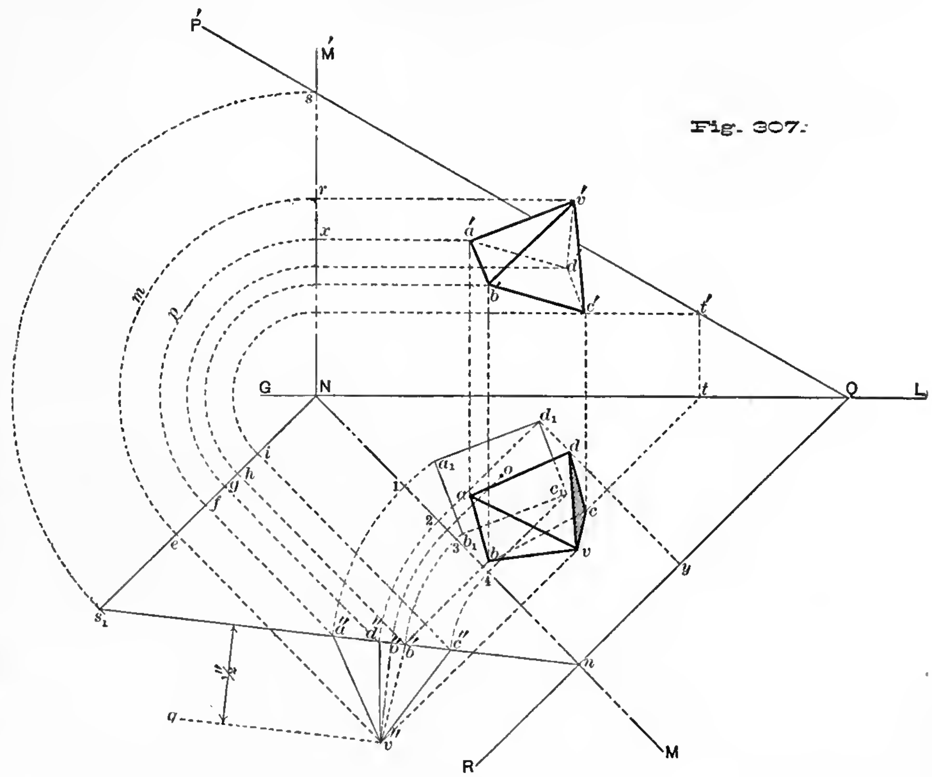
In the plane $P'QP$ assume the horizontal $t'c'$, $tc$, at the assigned height, and on it take some point $cc'$ for the lowest corner. By prolonging $tc$ we get $c''$ as the auxiliary view of $c$; $nc''$ then equals the real distance of $c$ from $RQ$, and, used as a radius for arc $c''4$, leads to $c_1$—the position of $c$ after rabatment about $RQ$. Draw $c_1b_1$ at the prescribed angle to $RQ$ and complete the rectangle $a_1b_1c_1d_1$ with sides in the assigned ratio.

Any corner, as $d$, of the plan is then found by projecting $d_1$ upon $nN$, thence by an arc (centre $n$) to $d''$, from which a parallel to $RQ$ meets $d_1y$ (the path of rotation) at $d$.

The elevations of the corners are at heights above G. L. equal to the distances of $a''$, $b''$, etc., from $MN$. This is indicated for $a'$ by $a''f$, the arc $fpx$, and the line $xa'$.

If we assume that the rectangle we have been considering is the base of a pyramid $\frac{1}{2}''$ high we have merely to erect at $o''$ (derived from $o$—the centre of the revolved rectangle) a perpendicular $o''v''$ of the prescribed altitude, and join $v''$ with the corners of the base. Also find $v$ and $v'$ by the process just outlined.

Fig. 307.



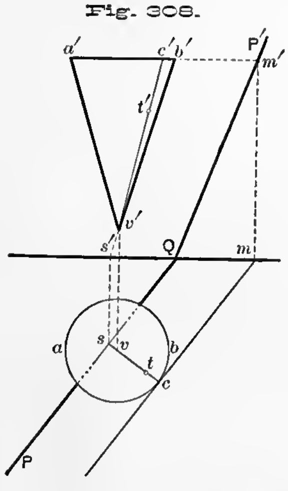

By turning the illustration so that $MN$ will be horizontal, the student will find that by looking in the direction $QR$, the auxiliary view $v.''a''b''d''c''$ will appear as the ordinary elevation, and the rotation from H to the space-position will be somewhat more clearly seen.

454. A plane and a surface are *tangent* to each other if they have a common point through which, if an auxiliary plane be passed, the line cut by it from the plane will be tangent to the curve which it cuts from the surface.

455. *Tangent planes to developable surfaces*, a few problems in which are next given, are solved by means of the principles stated in Arts. 368–376, which should be reviewed at this point.

456. *A plane, tangent to a cone at a given point*, will be determined by (a) the element through the point, and (b) a tangent to the base at the extremity of the element.

Fig. 308.

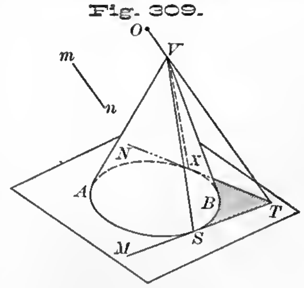



Let $v.'a'b'$, $v.ab$, (Fig. 308) be an inverted cone; $tt'$ the point at which the plane is to be tangent.

Draw the element $vc$, $v'c'$, containing the given point. Its h. t. is $s$, through which draw $PQ$ parallel to $cm$—the latter a tangent to the base at the extremity of the element. Join $Q$ with $m'$, the v. t. of the tangent $cm$ and we have in $PQP'$ the plane sought.

457. *A plane, tangent to a cone and containing a given point in space*, would be determined (a) by the line joining the given point with the vertex of the cone, and (b) by either tangent that could be drawn to the base from the trace of the first line upon its plane.

Fig. 309.




In Fig. 309 we have an oblique projection of this case, the orthographic being left for the student. $V.AB$ is the given cone; $O$ the given point.

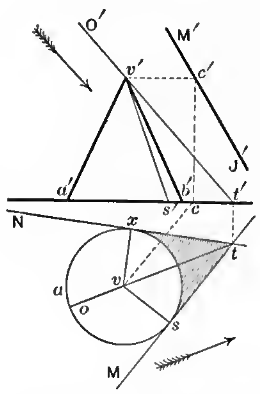
458. *A plane, tangent to a cone and parallel to a given line*, would be determined (a) by a line drawn through the vertex of the cone and parallel to the given line, and

(b) by either tangent (as in the last problem) drawn to the cone's base from the trace of the first line upon its plane.

In Fig. 309, were $m\,n$ the line parallel to which a tangent plane was required, then $V\,T$, parallel to $m\,n$, would with either $T\,M$ or $T\,N$ determine a plane fulfilling the conditions.

Were $m\,n$ the direction of the sun's rays, either tangent plane would be a *plane of rays*, and its line of contact would be an *element of shade.* The area between the tangents $T\,S$ and $T\,X$ (Fig. 309) would be cut off from the light by the cone and would therefore be the *shadow* of the latter. This case is illustrated orthographically in Fig. 310.

Fig. 310.



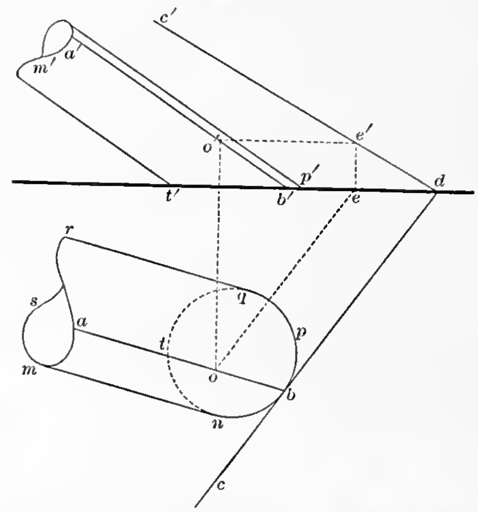

Through the vertex draw the ray $v\,t$, $v'\,t'$, parallel to the direction of light indicated by the arrows. Tangents $t\,s$, $t\,x$, from $t$—the h. t. of such ray—are the H-traces of the two possible tangent planes.

To find the vertical trace of plane $M\,t$ we may draw through the vertex $v$ a "horizontal" of the plane (300). Its projections are $v\,c$, $v'\,c'$, and its v. t. is $c'$, through which the vertical trace $M'\,J'$ would be drawn to meet $M\,t$ on the ground line. Similarly for the v. t. of the plane $N\,x\,t$.

459. *A plane, tangent to a cylinder at a given point.* Art. 376 again indicates the method of solution.

With $o\,o'$ (Fig. 311) as the given point, draw the element $a\,b$ through it, and the tangent $c\,d$ at its extremity. The latter is the h. t. of the required plane, and $d$ is one point of the v. t. Since the v. t. of the element $a\,b$, $a'\,b'$, is too remote we resort to a "horizontal" of the plane and through the point. This ($o\,e$, $o'\,e'$) has its v. t. at $e'$, which joins with $d$ for the v. t. required.

Fig. 311.

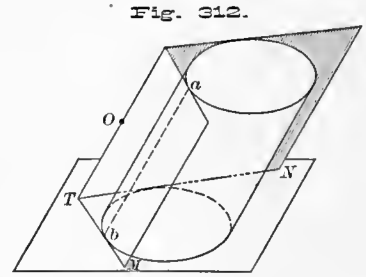



The oblique cylinder with circular base (right section elliptical) is used merely for convenience, to illustrate a general solution.

The other possible tangent to the base, parallel to $c\,d$, would be the h. t. of the other solution.

460. *A plane, tangent to a cylinder and containing a point exterior to it*, would be determined (a) by a line through the point parallel to the axis of the cylinder, and (b) by either tangent to the base of the cylinder from the trace of the first line upon the plane of the base. This is shown pictorially in Fig. 312, where $O$ is the given point; $O\,T$ the parallel to the axis; $T$, the trace of $O\,T$ on the plane of the base; $T\,M$ and $T\,N$ the tangents, either of which—with $O\,T$—determines a plane fulfilling the conditions.

Fig. 312.

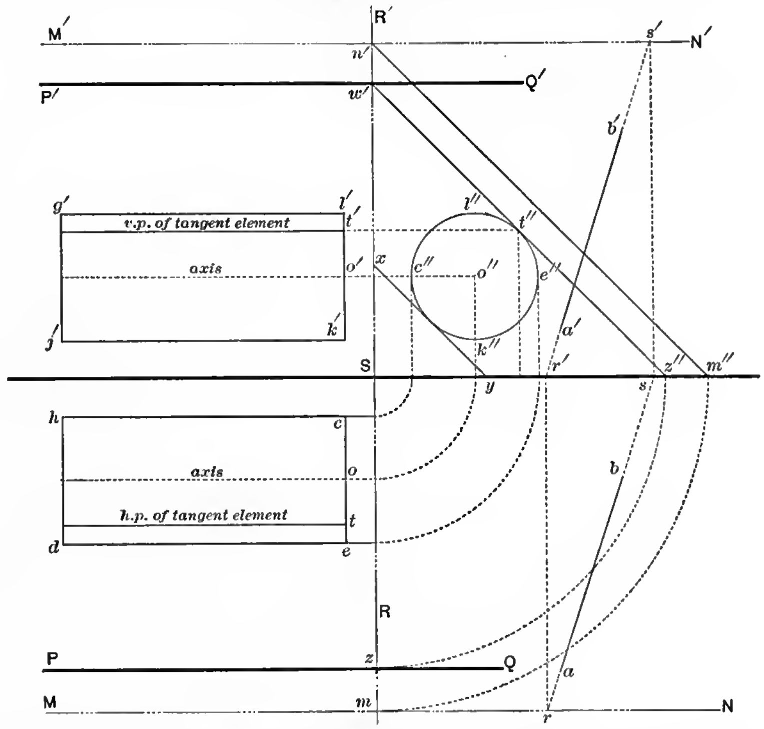



461. *A plane, tangent to a cylinder and parallel to a given line*, is determined most readily by making it parallel to a plane which can be passed, by Art. 315, through the given line and parallel to the axis of the cylinder. Since in Fig. 313 the cylinder is taken parallel to both H and V, a plane through the given line $a\,b$, $a'\,b'$, and parallel to the axis must therefore have traces ($M\,N$, $M'\,N'$) parallel to G. L.

$R\,S\,R'$ is *any* auxiliary, profile plane. It cuts a line from the $M\,N$-plane which, revolved, is

seen at $n'm''$. The profile view of the cylinder appears at $c''l''e''k''$, tangent to which draw $w'z''$ and $xy$, each parallel to $n'm''$, for the profile view of the required planes. $P'Q'$ and $PQ$ are then the traces of one plane fulfilling the requirements. The traces of the other are omitted, but would be similarly found.

Fig. 313.




462. *To pass a plane tangent to a developable helicoid at a given poin.* From the properties of this surface, discussed in Arts. 346 and 420, and of its base—the involute—treated in Arts. 186 and 187, a tangent plane meeting the assigned conditions can be as readily constructed as in the case of the cone. The element through the point, and a tangent to the involute at the extremity of the element, will determine it.

As in the case of the cone, all tangent planes to this surface will make the same angle with the axis; that is, the surface is of *uniform declivity*.

463. *A plane, tangent to a developable helicoid and containing a given exterior point*, may be determined by making the given point the vertex of a cone of the same declivity as the helicoid, the bases of both surfaces to be in the same plane. A tangent to both bases will then be one trace of a common tangent plane, provided that the elements drawn from the points of contact lie on the same side of such tangent. The v. t. of either element of contact will suffice for the completion of the solution.

464. *A plane, parallel to a given line and tangent to a developable helieoid*, will be *parallel* to a plane containing the given line and tangent to a cone whose vertex is on the line and whose declivity is that of the helicoid.

### WARPED SURFACES.—THEIR PROJECTIONS, TANGENT PLANES AND RACCORDMENT.

465. In Arts. 348–361 the principal warped surfaces are defined and illustrated, and some of their properties mentioned.

From the fact that but three conditions may be imposed upon a moving straight line we find that although there may be an infinite number of warped surfaces they may all be classified in the two following groups:

(a) Those whose generatrix moves upon three given lines, either straight or curved.

(b) Those having but two linear directrices but whose generatrix is, in its various positions, parallel to the elements of a cone, and, in particular, to a *plane*, regarding the latter as a special case of the cone. Such cone is called a *cone director*, and its limiting case a *plane director*.

The above classes are not to be understood as mutually exclusive, as any warped surface may be defined so as to belong to either.

466. *Warped surfaces with three linear directrices.* The element through an assumed point on one directrix may be found by making the point the vertex of a surface having either of the other lines as a directrix. This auxiliary surface—which will be either *plane* or *conical*—will be pierced by the third directrix in one or more points, through which the desired line(s) will pass.

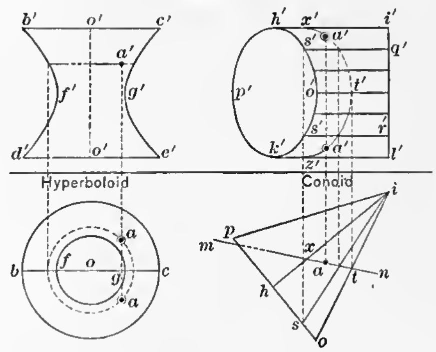
467. *Warped surfaces having two directrices and a cone (or plane) director.* To find an element through an assumed point of either directrix make such point the vertex of a cone similar to the cone director. The element sought will be the line joining the vertex with the intersection of the auxiliary cone with the second directrix. With a *plane* director an element would be found by passing through the given point a plane parallel to the plane *director*, noting its intersection with the other directrix, and joining it with the given point.

468. *Having given one projection of a point on a warped surface, to find the other.* The same general method applies as for developable surfaces, viz., draw on the surface a line passing through the given projection; then project from the given projection upon the other view of this auxiliary line.

This is illustrated in Fig. 314 by a *warped hyperboloid of revolution;* also by a *conoid*, a warped surface of *transposition.*

(a) *The hyperboloid.* This surface, represented by straight lines in Fig. 77, is shown in Fig. 314 as generated by revolving an hyperbola, $c'g'e'$, about its conjugate axis $o'o'$. Then, as for any other surface of revolution, obtain a circular section through the given projection $a'$, show the same *as a circle* on the plan, and project to it from $a'$, getting the two solutions.

Fig. 314.




Were the *plan* the *given* projection we would draw through it a tangent to the plan of the "gorge" circle as in Fig. 77, get the elevation of the same line as in Art. 116, and project up to it. Or we might, in Fig. 314, carry the plan $a$, by a circular arc, to the meridian plane $b\,c$, then project up to $c'g'c'$, the v. p. of the meridian curve, giving the *level* upon which to project $a$. (Further reference articles: 348, 350, 351.)

(b) *The conoid.* This surface, already alluded to in Arts. 354 and 355, is represented in Fig. 314 by its base $o\,p-o'k'p'h'$, its rectilinear directrix $i-i'l'$, and a few elements, the horizontal plane being the plane director.

Through $a$, the *given* projection, pass a vertical plane $m\,n$ at *any* angle to V. This cuts the element $o\,i$ at $t$, which projects to $t'$ in elevation. Similarly obtain $x'$, $z'$ and other points, and through them draw the curve $x'\,t'\,z'$, upon which $a$ projects in the two solutions.

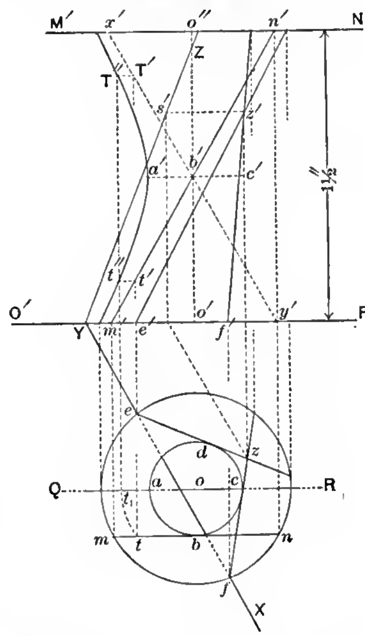
469. *Tangent planes to warped surfaces.* Before dealing with any particular case of tangency to a warped surface attention is again called to the fact, stated in Art. 377, that a plane, tangent to a warped surface at a given point, must be, in general, a secant plane elsewhere; for, by the law of generation of a warped surface, consecutive elements can not lie in the same plane, hence a tangent plane, of which an element is always one of the determining lines, must cut the adjacent elements.

From this it follows that any plane containing an element of a warped surface will (if not parallel to the elements, which it might be for any surface having a plane director) be tangent to the surface at some point of the element. For the element will be intersected somewhere by the curve cut from the surface by the plane. At such intersection a tangent line to the curve would, by Art. 454, lie in the tangent plane to the surface; and, by Art. 374, the element through that point would be the other determining line of a tangent plane.

*Point* and not *line* contact is thus seen to be the rule in the tangency of a plane to a warped surface; although, with surfaces like the conoid of Fig. 314, a plane parallel to the plane director and tangent at either $x'$ or $z'$ would be tangent all along the element through the point.

470. *The warped hyperboloid of revolution; projections and tangent plane.* Let the surface have a minimum diameter of $\frac{1}{2}''$; inclination of elements to H, 60°; height of surface, $1\frac{1}{2}''$.

Fig. 315.




*The projections.* In Fig. 315 draw two limiting planes, $M'N$ and $O'P'$, $1\frac{1}{2}''$ apart. Take $o$–$o'o''$ for the axis, and draw $a\,b\,c\,d$, $\frac{1}{2}''$ in diameter, for the plan of the gorge circle. Its elevation is $a'c'$, midway between the upper and lower bases. Through $b'$ a line $m'n'$, at 60° to H, is the elevation of an element parallel to V. Its plan $m\,n$ is next drawn, tangent (in *projection*, not in space) to the gorge circle. The circle through $m$ and $n$ completes the plan of the surface.

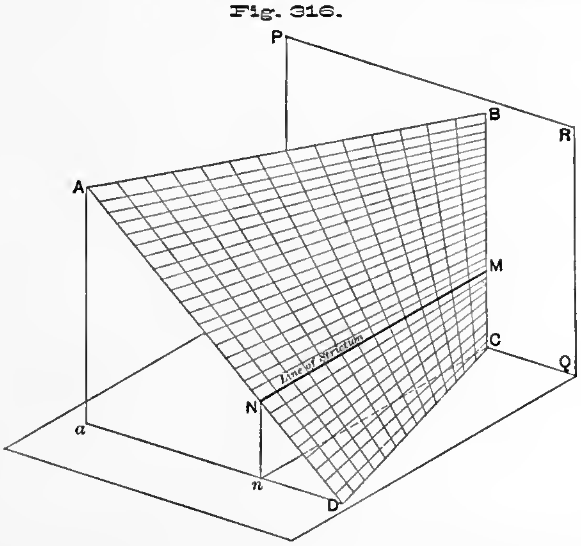
It will be noticed that the line $x'y'$ has the same plan as $m'n'$ and fulfills the same conditions; that is, its points are all equally distant from the axis; its rotation about $o'o''$ would therefore result in the generation of the same surface.

The elevation may be completed like Fig. 77 by drawing more elements, or like Fig. 314, by assuming points of the surface, which—like $t'$ and $T'$—have a common plan $t$, carrying them by arcs to the meridian plane $Q\,R$, and thence projecting to the *levels* of the elevations of the same points, getting $t''$ and $T''$ on the hyperbolic contour, or *meridian section.*

*The tangent plane.* The hyperboloid being one of the two possible doubly-ruled warped surfaces (Art. 350) its tangent plane at any point $z$ is most readily found by drawing through it the elements $e\,z$ and $f\,z$, and finding their plane $X\,Y\,Z$; although *one* element, together with the tangent to the "parallel" or horizontal circle through $z$, would suffice to determine it. (Art. 340.)

471. *The hyperbolic paraboloid; projections, also traces of a tangent plane.* Fig. 213 reappears in Fig. 316 for convenient reference, although the student should review Arts. 349–353 at this point.

Fig. 316.




(a) In accordance with the definition "having two straight directrices and a plane director" we see that the surface could be generated by moving $B\,C$ upon $A\,B$ and $D\,C$, keeping it always parallel to $P\,R\,Q$, as a plane director; or by moving $A\,B$ upon $A\,D$ and $B\,C$, keeping it parallel to the horizontal plane as a plane director.

(b) The elements of one set would be the sections of the surface by planes parallel to the plane director; hence they divide the elements of the other set proportionally. We might, therefore, obtain elements of an hyperbolic paraboloid by simply dividing two non-plane right lines into proportional parts and joining corresponding points. The plane director would be parallel to any pair of such elements, and would be found by Art. 315.

(c) $CQ$ is the line of intersection of the plane directors; $BC$ and $NM$ are the two elements—one of each set—whose directions are perpendicular to $CQ$; their intersection $M$ is called the *vertex* of the surface, and a parallel to $CQ$ through $M$ would be its *axis*.

The surface is divided symmetrically by two mutually perpendicular planes, called *principal diametric planes*, which contain the axis and bisect the angles between the elements meeting at the vertex.

When the plane directors are mutually perpendicular, as in the figure, the surface is called *right* or *isosceles;* otherwise it would be *oblique* or *scalene.*

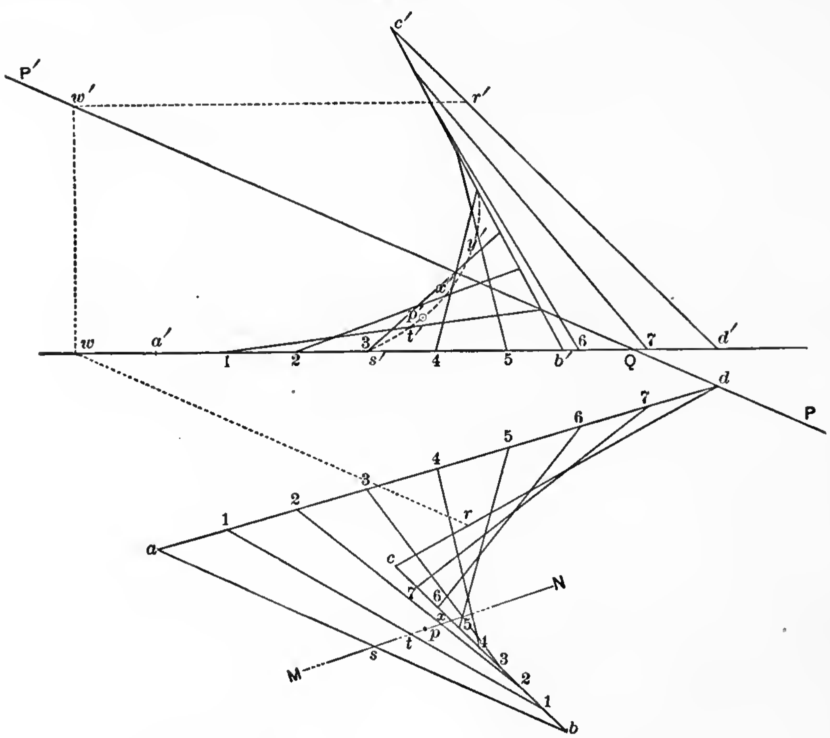
(d) If the elements parallel to $PRQ$ are projected upon it their projections will cross at the point $M$, which is the projection of $NM$, that element of the other set which is also the common perpendicular of the first set. Such point $M$ is called a *point of concourse.* For the horizontal set of elements the point $C$ is similarly a point of concourse.

As consecutive elements are non-plane, while all of one set are parallel to a plane director, we see that while no element can intersect another of the same set it must meet all of the other set.

(e) *The tangent plane at any point,* as $N$ (Fig. 316), would be determined by the elements $AD$ and $MN$ passing through the point. (Art. 374).

(f) *The surface in projection.* In Fig. 317 let $abcd$, $a'b'c'd'$, be a warped quadrilateral analogous to Fig. 210. $AB$ and $CD$ can then represent a pair of elements of one generation; $AD$ and $BC$ of the other.

Fig. 317.




Let $p$ be the plan of a point on the surface. To find its elevation draw a series of elements by the method indicated in the latter part of Case (b) of this article, and cut them by a secant plane containing $p$. Thus, divide $AD$ and $BC$ into eight equal parts, for example, and join the like numbered points of division. Cut these elements by any auxiliary vertical plane $MN$, containing $p$. The curve obtained is shown in elevation at $s't'x'y'$, upon which $p$ projects at $p'$.

$P'QP$ is the plane director for the elements indicated, and is determined by $cd$ and a parallel to $ab$ through some point $r$ of $cd$. The other plane director (not shown) would be similarly found.

(g) *An element containing a known point, as $pp'$, of the surface.* Through the point pass a plane (by Art. 316) parallel to the plane director of either set; it will cut either directrix of that set in a point which joins with the given point for the element sought.

472. In arches having warped soffits, those surfaces of voussoirs which have to be normal to the soffit are hyperbolic paraboloids, unless the "twist" of the theoretical normal surface from its

limiting plane is so slight that a plane bed may be substituted without imperilling stability. (See Art. 475.)

Another and more readily observed application of this surface is in the "cow-catchers" of locomotives, which are usually made of two hyperbolic paraboloids, symmetrically placed with reference to a vertical, longitudinally-central plane of the engine, the "elements" being bars, either parallel to such vertical plane as a plane director or else horizontal.

473. *A conoidal surface.* Conoidal surfaces are defined in Art. 354, and several of them illustrated in Figs. 214, 215 and 217.

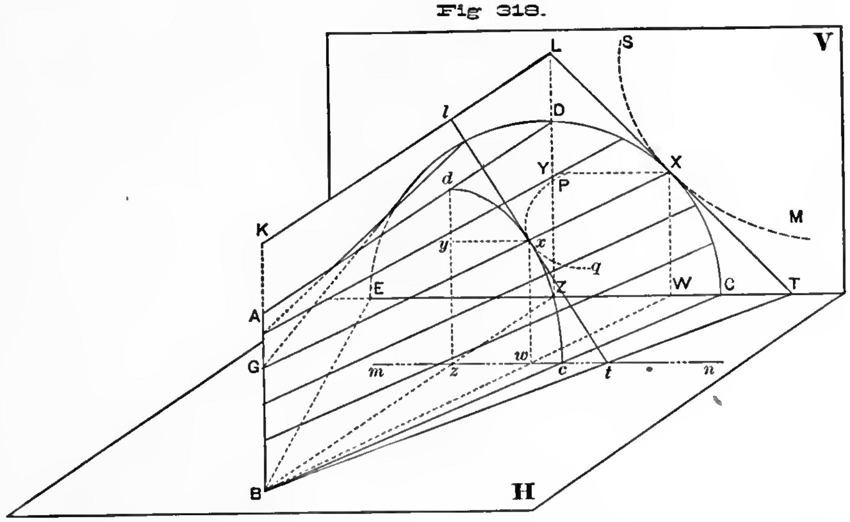
(a) *The orthographic projections* of the *cono-cuneus* of Wallis are shown in Fig. 314, and sufficiently treated in Art. 468 (b).

(b) *To get a tangent plane to the cono-cuneus.* To solve without resorting to an auxiliary surface we need to know *the nature of plane sections parallel to the base.* That they are *ellipses* may be thus established: In Fig. 318 let H be the plane director; $CDE$ the circular base; $AB$ the right line directrix, and $cxd$ a plane section parallel to the base. The semi-axes $dz$ and $DZ$ are equal; and from the similar triangles $Bcz$ and $BCZ$, and the equality of $xy$ to $wz$ and of $XY$ to $WZ$, we have $xy : cz :: XY : CZ$, a characteristic of ellipses having equal major (or minor) axes.

(c) A tangent plane at some point $x$ would then be determined *directly* by the *element* $GX$ and the *tangent* $lt$ to the elliptical arc $dxc$; or, *indirectly*, by obtaining $lt$ as an element of an auxiliary surface, as described in the next article.

474. *Raccordment.** We have seen (Art. 469) that a tangent *plane* to a warped surface has, usually, *point* and not *line* contact with it; it is, however, possible for two *warped* surfaces to be mutually tangent at every point of a common element. The surfaces are then said to *raccord* along that element.

Fig 318.




Raccordment exists whenever two warped surfaces have a common element and a common tangent plane at each of *any three* points that may be taken upon it; a secant plane at either of such three points will then cut from the surfaces lines which will be tangent to each other, and which could be used as directrices of their respective surfaces. *Three* points of common tangency must be established, since, by Art. 343, three conditions are imposed on the generatrix of a warped surface, and each of them must be consistent with tangency to ensure raccordment.

Although the conoid is employed in illustration of the foregoing the conclusions are perfectly general, and applicable to all warped surfaces.

In Fig. 318 regard $AB$, $dxc$ and $DXC$ as sections of the conoid $BA\text{-}CDE$ by three *parallel* planes. Draw tangents to these sections at the points where they are met by some element, as $GX$. $AB$, being a straight section, may be regarded as its own tangent (Art. 370), while $tl$

*On account of the utility of an auxiliary raccording surface in passing tangent planes to warped surfaces, this topic is presented at this point in order to apply it to the warped surfaces yet to be treated.

and $TL$ are obviously tangent in the ordinary sense; and as all three are parallel to the same plane they will, if used as directrices for a warped surface, give an hyperbolic paraboloid (Art. 349) which would evidently have a tangent plane in common with the conoid at each of the three points $G$, $x$, and $X$; it would therefore raccord with the conoid along the element $GX$.

Since $TB$ is the horizontal trace of the paraboloid $BKLT$ we may obtain $lt$ by joining $x$ with $t$ which is the intersection of $TB$ by the trace $mn$ of an auxiliary plane parallel to V.

Were the section $dxc$ and its tangent not in a plane parallel to V we would obtain a raccording *warped hyperboloid* by using $AB$, $LT$ and the new $lt$ as directrices. (Art. 349.)

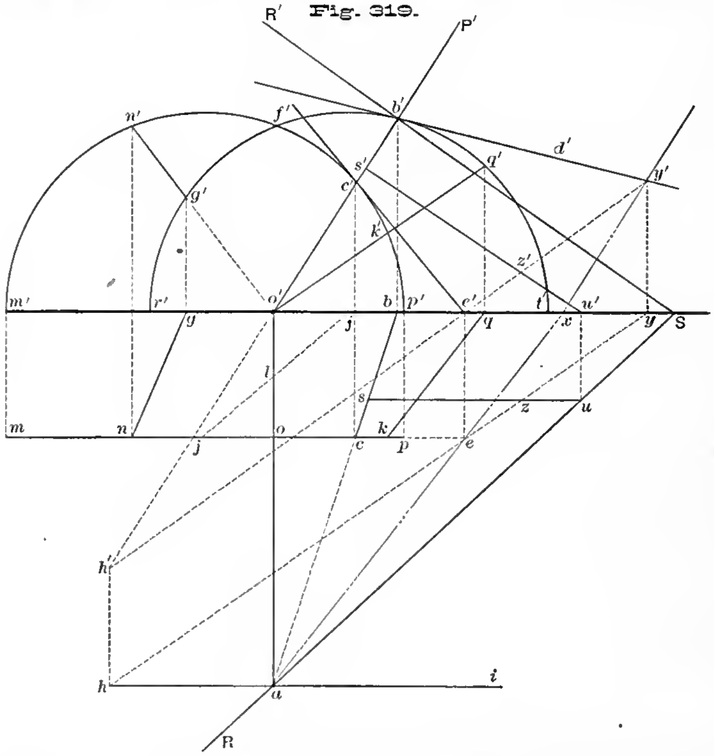
Since an infinite number of sets of parallel planes or of non-parallel planes could be passed through $G$, $x$ and $X$, in each of which a section of the conoid would—with its tangent—lie, it is possible to have an infinite number of hyperbolic paraboloids, or of warped hyperboloids, raccording with a given warped surface along the same element.

In a case like that of Fig. 318, if we regard the plane director, it is only necessary to show that there are *two* tangent planes and a plane director in common, in order to have the usual number of conditions on the generatrix of each surface, and at the same time ensure raccordment.

Let $Pxq$ and $SXM$ be *any* curves in the planes cutting $dxc$ and $DXC$ from the conoid, and having tangents in common with them, at $x$ and $X$; a surface generated with these new curves as directrices, and having the same plane director as the paraboloid $BKLT$ and the conoid, would raccord with them both, along the element $xX$, as would also be the case if the third directrix, $AB$, were substituted for the plane director.

475. *Normal hyperbolic paraboloid.* Were the surface $BKLT$ (Fig. 318) rotated 90° on $GX$ as an axis, the lines $AB$, $lt$ and $LT$ would become *normals* to the sections to which they are now tangent, and the paraboloid would be *normal* to the conoid along the entire element. A hyperbolic paraboloid that shall be normal to any warped surface along a given element may therefore be found as readily as a raccording surface.

Fig. 319.




476. *The Corne de Vache* (Cow's Horn). To represent this surface in conformity with the definition in Art. 361, and as there illustrated, draw (Fig. 319) the semi-circle $m'n'p'$, $mp$, parallel to V, and the *equal* semi-circle, $r'b't'$, *in* V. Join their centres by the line $jj$, and through its middle point, $l$, draw $al$, $o'$, perpendicular to V, for the straight directrix. Any plane containing $al$, as $ao'P'$, for example, will be perpendicular to V, and will cut the curved directrices in points, as $bb'$, $cc'$, which, joined, will give an element of the surface. Similarly, the plane $ao'q'$ gives the element $k'q'$, $kq$. (See Art. 466.) The element at $f'$ is obviously horizontal.

*A tangent plane,* at any point, $ss'$. Draw $b'c'$, $bca$, the element containing the point. An auxiliary hyperbolic paraboloid, raccording with the given surface along $b'o'$, $bca$, may

be generated by moving the latter upon $b'd'$—which is a tangent *in* V, and upon two other lines that are tangent, at points of the assumed element, to sections made with the surface by planes *parallel* to V. (Art. 349.)

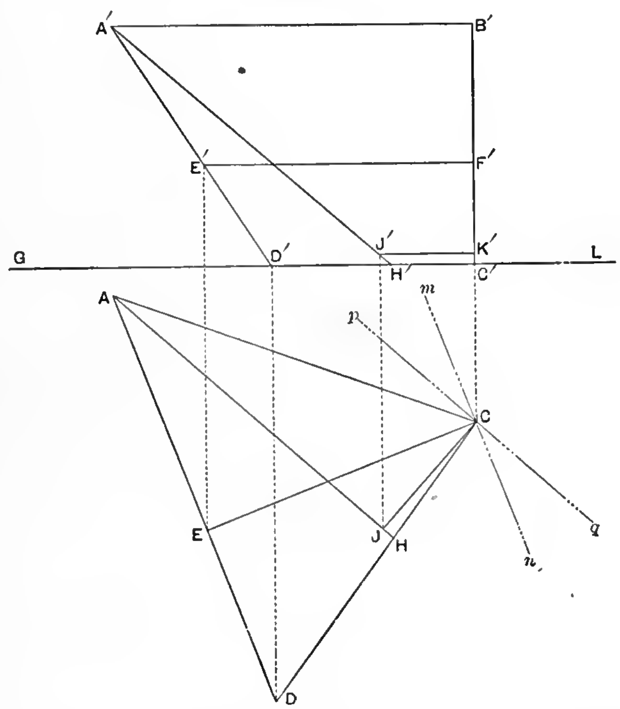
In $c'e'$, $ce$, we have one such tangent, and in $h'P'$, $hi$, another, the plane $ao'P'$ being a tangent plane at $a$ because determined by an element $ba$, $b'o'$, and a straight directrix $o'a$, each of which possesses the distinctive property of a tangent. (Art. 374.)

A second element, $h'y'$, $hy$, of the same set as $bc$, is found by Art. 466 thus: Pass the plane $axy'$ containing one directrix, $hi$, $h'P'$, and some point, as $e$, of another. The v. t. $(xy')$ will be parallel to $o'P'$, since one of the determining lines is a V-parallel. This plane cuts the third directrix at $y'$, whence $y'e'$ and $ye$ follow, for the line sought.

Lastly, draw through $s$ a parallel to the ground line, for the plan of an element parallel to V and on the paraboloid. It meets $ey$ at $z$, which projects to $z'$, giving $s'z'$ for the v. p. of the element. $RSR'$ is then the desired tangent plane at $ss'$, being determined by two lines, $s'o'$, $sa$, and $s'z'$, $sz$, each an element of one set in an auxiliary raccording surface.

477. *The conoid of Plücker.* (Cayley's *cylindroid.*) This surface plays the same role in the determination of the combined effect of two simultaneous twists or wrenches on a solid body, as the parallelogram of forces in finding the resultant of two forces acting simultaneously upon a point in their plane.

Fig. 320.




(a) Referring to Figs. 213 and 215 and the first definition of Art. 356, we get elements of this surface as lines of striction of hyperbolic paraboloids thus: In Fig. 320, which is an orthographic projection of the $ABCD$ of Fig. 213, let $CA$ $(B'A')$ and $CD$, $C'D'$, be horizontal lines having $C'B'$ for their common perpendicular; then with $mn$ as a plane director we have $CE$ (perpendicular to $mn$) for the line of striction of the set of elements parallel to $mn$, and $E'F'$ for its elevation. With $pq$ as the h. t. of a plane director, $AH$ will be the plan of the element through $A$, and $CJ$ $(J'K')$ the corresponding line of striction, and therefore another element of the conoid under construction. Others may be similarly found.

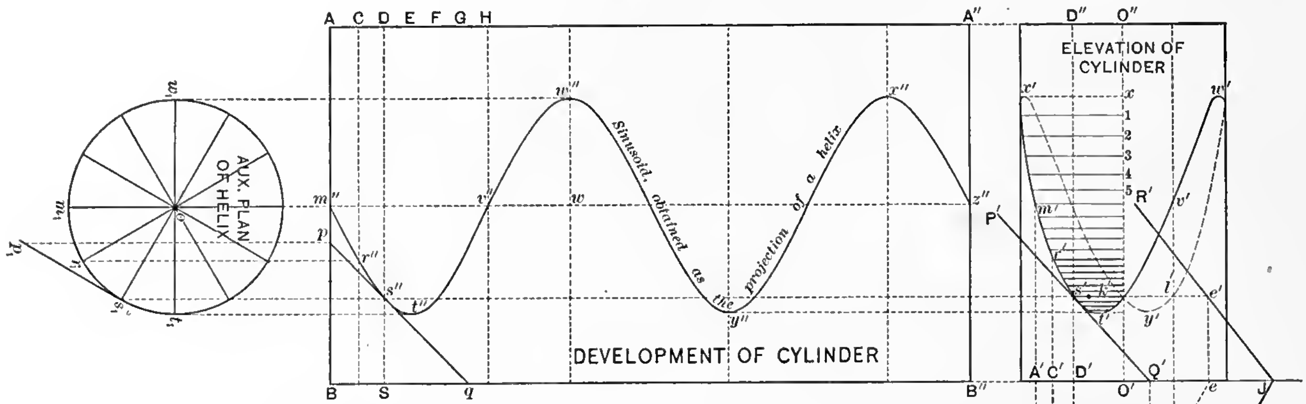
(b) The conoid of Plücker may be obtained by taking H for its plane director; for its curved directrix an elliptical section of a vertical right cylinder of circular base; for its straight directrix the element of the cylinder at either extremity of the major axis of the elliptical directrix. Any circular cylinder having for its *axis* the straight directrix just indicated will cut the surface (assuming the elements to be indefinitely extended) in the double-curved directrix employed in Case (c).

(c) To represent the surface in accordance with the second definition of Art. 356 draw first a rectangle $ABCD$ (Fig. 321) as the development of a cylinder whose length is twice the wave length of the sinusoid.

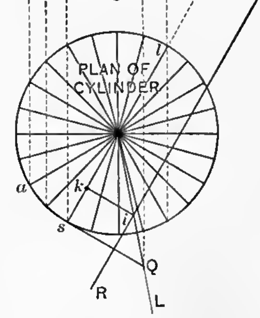
The fact that a helix projects as a sinusoid may be availed of, and the latter curve now ob-

tained by the usual method for the former. Thus, extend the central line, $m''z''$ of the development, to some point $o$, which use as the centre of a circle whose radius equals the amplitude $(w\,w'')$ of

Fig. 321.




each wave; then construct as a helix the curve $m''t''w''y''z''$, as in Art 120, making, however, two complete undulations.




Draw next the plan and elevation of the cylinder on whose development we have been working, and locate upon it the various elements previously represented. Thus, $a, A'm'$ represents $AB$ of the development, and upon it $m''$ projects at $m'$. $SD$ appears at $s, D'D''$, upon which $s'$ is found from $s''$. By connecting other points similarly obtained we get $m's't'w'y'x'$ for the curved directrix of the conoid. Horizontal lines through its points and the axis, as those through $x$, 1, 2, 3, etc., are then portions of elements. Their plans, extended to diameters, represent all of the surface included by the cylinder.

(d) *To draw a tangent*, as $s'Q'$, at any point $s'$ of the curved directrix, revert to the point $s''$. from which $s'$ was derived; draw $s''p$ (from $s_1p_1$) as a tangent to a helix (by Cases (b) and (c) of Art. 420); prolong $ps''$ to $q$; then on the plan make tangent $sQ$ equal sub-tangent $Sq$, and project $Q$ to $Q'$, when $Q's'$ will be the desired tangent.

(e) *The tangent plane* at any point of the curved directrix would be determined by the element and the tangent at that point to the curved directrix.

The tangent plane $RJR'$, at any random point of the surface, as $k$, is most readily found by means of an auxiliary hyperbolic paraboloid, *raccording* with the conoid along the element through the point. Its determining lines are $k\,i$ (plan parallel to $s\,Q$) and $s\,e$, $s'e'$. (See Art. 473.)

It is an interesting fact that all tangent planes to the Plücker conoid (and by Art. 469 every plane containing an element is a tangent plane) will cut it in ellipses.*

478. *The right helicoid.* This is a warped surface of the conoidal family, having a helix and its axis for directrices, and a plane director perpendicular to the axis. (Arts. 357 and 358.)

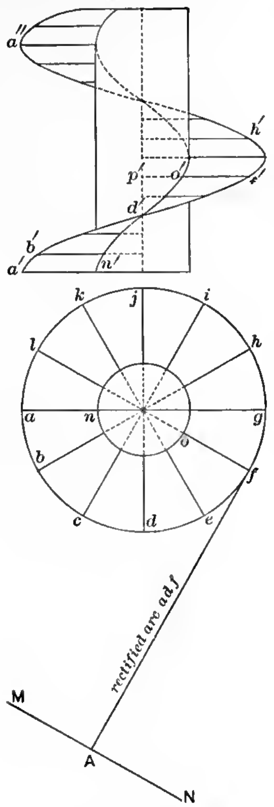
To project it orthographically draw a helix in the usual manner (Art. 120), and a series of horizontal lines through the axis and terminating on the curve. Thus, in Fig. 322 the helix $a'd'h'a''$ is the curved directrix, and the horizontal lines (radii, in plan) are the elements.

*A tangent plane* at some point $ff'$ is determined by (a) the element $f'o'$ containing the point,

* For the more recent developments of the Theory of Screws and applications of the conoid of Plücker see Gravelius' translation of Prof. Ball's work; Berlin, Georg Reimer, 1889.

and (b) by the tangent to the helix at $f'$. From Case (e) of Art. 420 we see that $fA$, the plan of such tangent, must equal the arc $fda$, and that $A$ is the trace of the tangent; $MN$, drawn parallel to the element $fo$, is then part of the h. t. of the tangent plane, since $fo$, $fo''$ is a horizontal (Art. 300) of the plane. The v. t. of the plane would pass through that of the element.

Fig. 322.



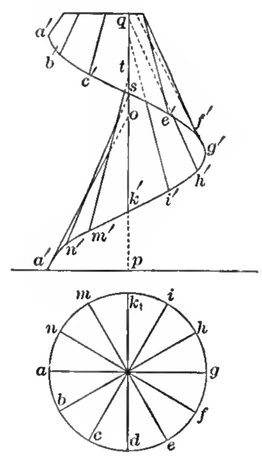

Were the point of desired tangency *not* on the helical directrix a separate helix would have to be constructed, containing the point.

Although the *pitch* (or *rise* in one revolution) may be the same for two helices, the one nearer the axis would obviously have the greater steepness or declivity.

479. *The oblique helicoid.* This warped surface (see Fig. 216) has a helix and its axis as directrices, the elements making a constant *acute* angle with the latter; hence it comes in the class of surfaces having a *cone* director; a *right* cone also, since the obliquity is constant.

To project it orthographically construct a helix $a'k'g'$ (Fig. 323), as in Art. 120; draw an element, $a'o$, $ag$, parallel to V and at the assigned inclination; then, since its extremities must ascend at the same rate, join $n'$, $m'$, $i'$, etc., with points $s$, $t$ and $q$, whose vertical heights above $o$ equal those of $n'$, $m'$ and $i'$ respectively above the level of $a'$.

Fig. 323.



The visible contour of the elevation will not be straight, but a curve, tangent to the projection of the elements; their *envelope*, in other words. (Art. 335.)

*A tangent plane* at any point would be found exactly as for a right helicoid, viz., with an *element*, and a *tangent to the helix* through the point.

Any section of this helicoid, by a plane perpendicular to the axis, will be a *spiral of Archimedes* (Art. 188). This can be shown by taking a series of elements whose plans make equal angles with each other, and carrying them parallel to V. Being then both parallel and equidistant, in space, it will be found that they will, if produced, cut a given horizontal plane at distances from the axis that are in arithmetical progression.

480. *The general cylindrical helicoid.* The right and oblique helicoids just described, and the developable helicoid of Arts. 346 and 420, are but special cases of the general cylindrical helicoidal surface, having a cone director, and two helical directrices lying on con-axial cylinders.

Fig. 324 illustrates such a helicoid, the helical directrices lying on cylinders of diameters $ag$ and $qn$ respectively, only the smaller being shown in elevation. The elements $bi$, $ck$, etc., are tangent both *in space* and *in plan* to the inner cylinder.

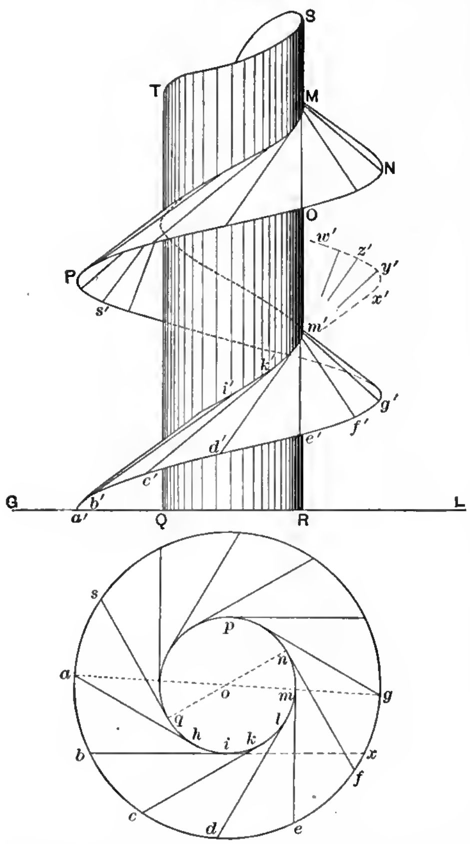
The shortest method of construction is to draw the plan circles, and with the larger construct the outer helix $a'd'g'$, of the given pitch and in the usual way (Art. 120); draw tangent $bi$, parallel to G. L., as the plan of the element parallel to V, and whose actual inclination to H may therefore be seen on the elevation; make such elevation ($b'i'$) at the desired angle to G. L. As $i'$ is then one point of the inner helical directrix we may draw the latter through it, making it of the same pitch as the outer helix; then $k$, $m$, etc., will project upon the inner helix at $k'm'$, to be joined with $c'$, $e'$, etc., for the elevations of elements.

If $bi$ be prolonged to $x$ the latter point will trace upon the the larger cylinder a helix iden-

tical in form with that traced by $b$, although not coinciding with it, except under the peculiar conditions described in the next article.

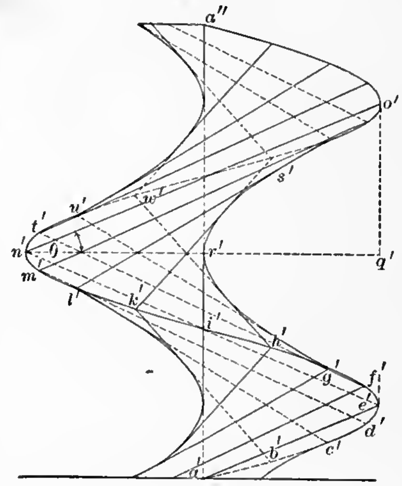
The portion $i x$, $i' x'$, of the element $b x$, will generate a nappe of the helicoid whose concavity is upward, but otherwise like the other. A portion of such generation is suggested at $x' y' w'$.

Fig. 324.



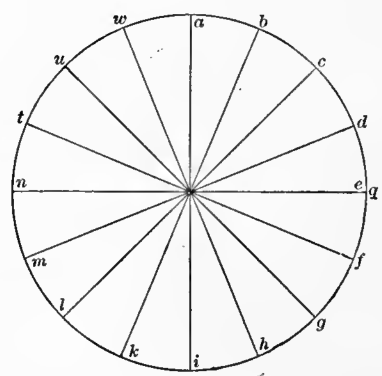

Were the generatrix $b x$, $b' x'$, to lengthen equally on each side of $i$, the nappes $M N O$ and $m' x' w'$ would evidently approach each other until finally they would intersect in a helix. Further elongation of $b x$ would simply result in additional helical intersections; in other words, a helicoidal surface, if indefinitely extended, will intersect itself in an infinite number of helices. To find one such helix pass a vertical plane through $b x$, $b' x'$; determine the curve in which this plane cuts the nappe $M N O P$ (extended) by joining the points in which the successive elements on that nappe meet the plane; then $b' x'$ will meet such curve in the point which generates the helical intersection of the nappes.

Fig. 325.



481. In Fig. 325 an interesting case of helical intersection is shown, the conditions being such that the outer helix is traced twice, it being the path of each extremity of the generatrix.

To obtain the surface in this form the diameter of the inner cylinder must reduce to zero, and the elements must intersect the axis at an angle $o' n' q'$, or $\theta$, whose tangent equals $o' q' \div n' q'$; that is, *half the pitch* divided by $2 r$.



Positions of the generatrix that differ by a semi-revolution, as $e' n'$ and $o' n'$, will intersect each other, as at $n'$, showing that the helical directrix is also the first of the helices in which the surface intersects itself.

The contour line $g' r' s'$ is called by Bardin a species of hyperbola, since it has $n' o'$ and $n' e'$ for asymptotes; but it is not coincident with a true hyperbola having the same vertex ($r'$) and asymptotes.

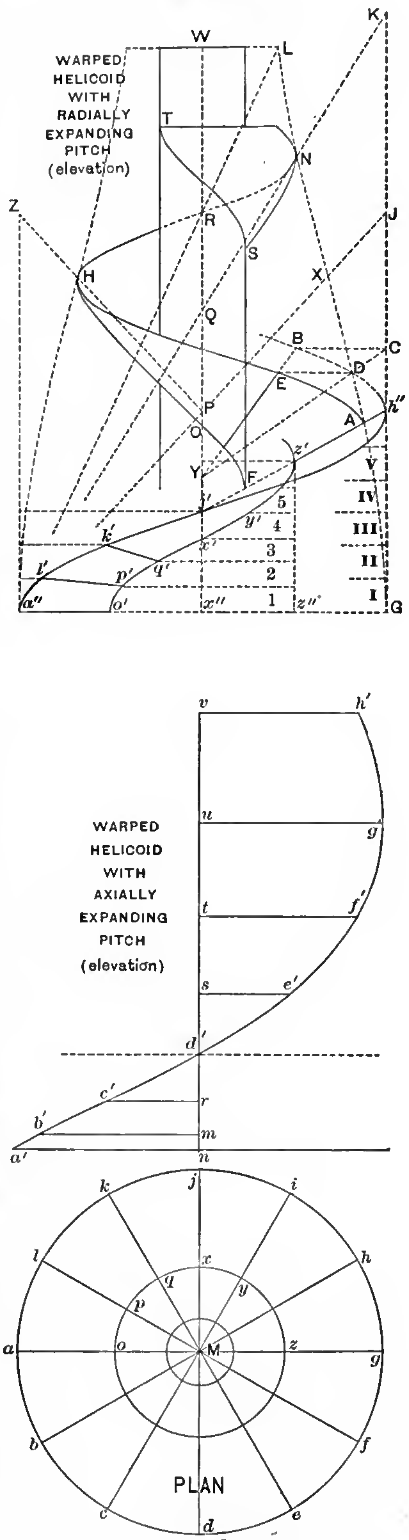
482. *Classification of helicoids of uniform pitch.* Let $r$ be the radius of the inner cylinder; $R$ that of the outer. Represent by $\theta$ the inclination of the generatrix to H, and let $\beta$ denote the like inclination of the inner helix. We then have the following cases:—

(a) *General warped helicoid*, when $\theta$ is greater or less than $\beta$, and $r$ is finite.

(b) *General warped helicoid*, (plane director), with $r$ finite and $\theta = 0$.

(c) *Developable helicoid,* whenever $\theta = \beta$, and $r$ is finite and either less than or equal to $R$.

Fig. 326.




(d) *Surface of triangular-threaded screw,* with $r = 0$, and $\theta$ either greater or less than $90°$.

(e) *Surface of square-threaded screw,* for $r = 0$ and $\theta = 90°$.

The student will obtain some interesting results by constructing case (a) with various values of $\theta$, contrasting in particular, the form in which the advancing half of the generatrix is lower than its point of tangency on the inner cylinder, with the opposite case of relative position. The meridian and right sections of the various helicoids also present some interesting features.

483. *Helicoids of radially expanding pitch.* These result from the combination of a *uniform* motion of *rotation* of a generatrix, with a *variable* motion of its points, *axially,* in such manner that the pitches of the helices described by the points are proportional to the distances of the latter from the axis. Such a helicoidal surface has been employed for the screw propeller on the theory that it would tend to counteract the centrifugal action of the water which the screw had set in motion.

Since each point of the generatrix traces a helix of uniform pitch, the intersection of the surface by any circular cylinder conaxial with it will be an ordinary helix.

484. *Construction of a helicoid of radially expanding pitch.* In the upper portion of Fig. 326 we have in $a''l'h''$ and $o'p' \ldots z'$ two of the helices on a surface of this kind, each obtained in the usual way; that is, for the former, by dividing the half-pitch, $h''G$, into the same number of equal parts (six) as the half plan $alk \ldots g$, and projecting $a$ to $a''$, $l$ to $l'$, etc., upon the dotted horizontals through the points of division. Similarly, divide $z'z''$, the semi-pitch of helix generated by $o'$, into the same number of equal parts by horizontal lines, upon which project $o$, $p$, $q$, etc., from the plan of the same helix.

The line $h''z'j'$ is the position of the generatrix $a''x''$ ($aM$) after the semi-rotation supposed, and $x''j'$ therefore the half-pitch of the point that travels along the axis. (It is assumed that the generating line may be of indefinite length, and merely has the two helices as directrices). In $a''o'$, $l'p'$, $k'q'$, etc., we have portions of the generating line (not *equal* portions) included between the helices.

In the figure, $j'x''$ is taken at one-half $h''G$, but any desired proportion may obviously be assumed.

Probably the most interesting special form of this surface is that called Holm's *conchoidal screw,* from the fact that by employing only that part of the surface which was generated by a fixed portion of the initial line, he included the helicoid within a surface of revolution whose meridian section was the "superior" branch of a conchoid. In the figure the surface as thus limited begins at $FA$, and is thus constructed:—

Make $j'P$, $PQ$, $QR$, each equal to $j'x''$. Also make $h''J$ and $JK$ equal to $h''G$; then, with $Z$ at the same level as $J$, we see that $PZ$ and $QK$ are positions of $j'h''$ differing from each other by a semi-revolution, and that if we rotate $PZ$ to $PJ$ and then make $j'A$, $PX$, $QN$, $RL$, each equal to the original length $a''x''$ $(=x''G)$, the curve $LN..DG$ will be a *conchoid*, whose revolution about $Wx''$ will give the new locus of the point $a''$.

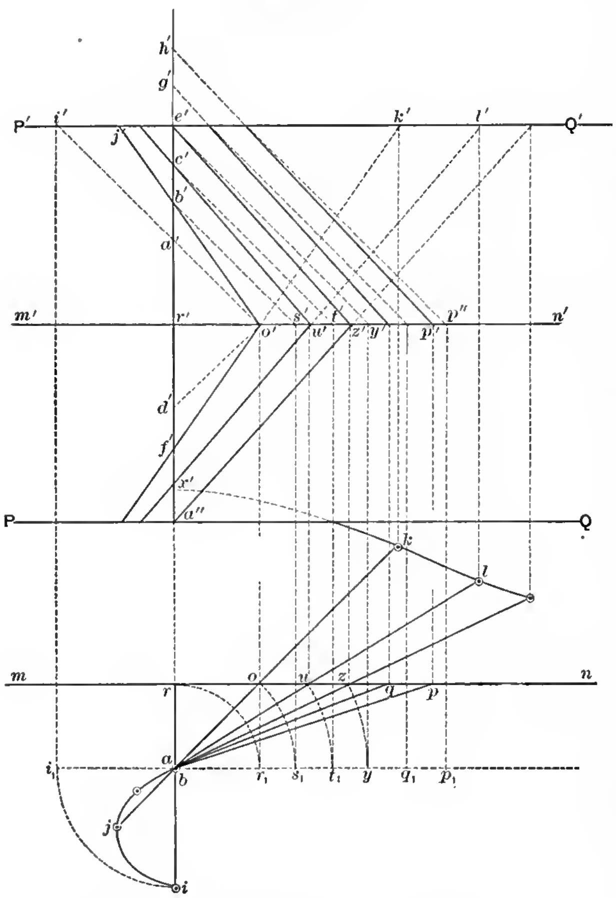
The determination of the *curve* $AEHN$, traced upon this new surface by the point $A$, is stated in the next article in the form of a general problem, and it need only be further remarked as to the *surface* in question, that as $Wx''$ is an asymptote to the conchoid, the helicoid will, at its limit, become a plane, tangent to a meridian plane of the surface.

485. *To determine the line of intersection of a helicoid with a con-axial surface of revolution.*

Illustrating from the upper elevation in Fig. 326, let $a''j'h''B$ be the *directrix*, and $YB$ any *element* of the helicoid.

Let $LNDG$ be the meridian section of a surface of revolution having the same axis, $Wx''$, as the helicoid. Rotate the element $YB$ to $YC$, when it and $LNDG$ are seen in true relation to one another, being parallel to $V$. They intersect at $D$, which counter-revolves to $E$ on the original position of the element.

Fig. 327.




The same process may be repeated with other elements until a sufficient number of points have been determined for the drawing of a fair curve.

486. *Helicoids of axially-expanding pitch.* If the generating line of a helicoid have a *uniform turning* motion about the axis, combined with a *varying rate* of motion in the direction of the axis, it is said to have an *axially expanding pitch.* With its generatrix *intersecting* the axis such a helicoid has been constructed for the acting surface of a screw propeller, with the idea that the water upon which it acted would be followed up by the elements which had set it in motion, at a rate in some degree approximating the accelerated movement of the receding mass.

487. *To draw a helicoid of axially expanding pitch.* In Fig. 326 the axis $nv$ is divided into parts $nm$, $mr$, etc., that are in some ratio to each other; in arithmetical progression in this case. Horizontal lines are then drawn through the points of division, upon which—as for the ordinary helix—the points $a$, $b$, $c$, $d$, etc., are projected from the plan to obtain the helix shown, the elements in plan making equal angles with each other.

Since the *ordinary* helix develops into a straight line, it is obvious that a helix of the kind under consideration will develop into a curve.

488. *The conchoidal hyperboloid of Catalan.* In accordance with the definition of Art. 359 draw any vertical line $a''e'$, $ab$ (Fig. 327) for one directrix; let the horizontal directrix be the line

$m'n'$, $mn$, and assume 45° for the inclination of the elements to the vertical directing line.

In $ar$ we have the plan of that element that meets $a''e'$ nearest to its middle point. Carry $r$ to $r_1$ whence project to $m'n'$ at $o'$, when $o'a'$, at 45° to the vertical, will be the element in revolved position. After counter-revolution its vertical projection becomes $a'r'$.

On the vertical directrix lay off, for convenience, *equal* spaces from $a'$, as $a'b'$, $b'c'$, etc. Since the elements are all equally inclined they will be parallel to each other if rotated till parallel to V; hence the dotted parallels through $a'$, $b' \ldots h'$ will represent a few in such position, their common plan being then $ap_1$. In counter-revolution $s_1$ reaches $o$, projects to $o'$ and joins with $b'$, for the space-position of $b's'$. Similarly $c't'$ becomes $c'u'$, $cu$; and from $y'$ we have $y$, then $z$ and $z'$.

By making $r'd'$ equal to $r'a'$, and working downward from $d'$ as we have upward from $a'$, a second surface of the same nature is formed, on the same directrices and with elements having projections coincident with those of the first surface. The two surfaces can best be distinguished from each other by the use of colors. The elements that are visible in front of $mn$ on the plan are on that portion of the surface which is visible above $m'n'$ in the elevation, and would be in the same color; while the part behind $mn$ and below $m'n'$ would represent the other visible portions, and would have the other color. The student should note, however, that the same part of one element is not visible in both views, in the latter case, but is in the former.

By prolonging an element, as $f'o'$, $ao$, to meet any limiting horizontal plane, $P'Q'$, we obtain a point $k'k$ of the *conchoidal* arc $lkx'$ (Art. 193) in which such plane would cut the surface. The other element $o'j'$ passing through $o'$ will meet the same plane in a point $j'$, which gives on the plan $ao$ a point $j$ of $aji$, a part of the other branch of the conchoid.

*A tangent plane* to the conchoidal hyperboloid at any point would most simply be determined by the *element* containing the point, and the *tangent* to a conchoidal section through the point, a method for drawing which has been given in Art. 195.

489. *The cylindroid of Frezier.* This surface may be readily constructed by the student without other illustration or definition than that already given in Art. 360. In Fig. 220 $abcd$ is the plan of a cylindroid, and the oblique figure is an enlarged elevation of the same.

*A tangent plane* at any given point would most conveniently be determined by means of an auxiliary hyperbolic paraboloid raccording with the cylindroid along the element through the point, V being the plane director; while the directrices would be the tangents to the elliptical bases at the extremities of the element.

490. Either (a) *through an exterior point*, or (b) *parallel to a given line*, it is, in general, possible to pass an *infinite* number of *tangent planes to a warped surface.*

In the former case they would all be tangent to a cone having the given point for its vertex, and for elements the tangents drawn through the point to the curves cut from the surface by planes containing the point, since any such tangent would—with the element through the point of tangency—determine a tangent plane.

In the latter case they would be the possible tangent planes to a cylinder whose elements are tangent to the curves in which the surface is intersected by a system of planes parallel to the line.

491. If a given line be prolonged to meet a warped surface, the element through their intersection will—with the given line—determine *a tangent plane to the surface and containing the line.* (Art. 469).

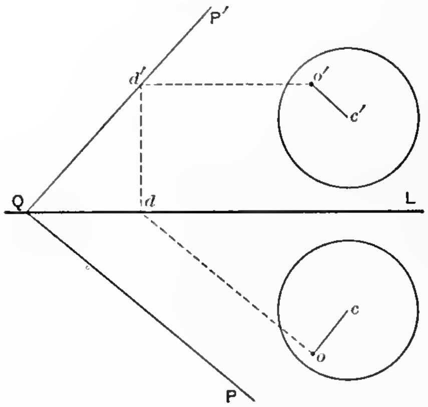
If the line can meet more than one element, whether because the surface is doubly-ruled or by reason of intersecting the surface more than once, or when both of these conditions exist simultaneously, each element that it meets would—with the given line—determine a tangent plane.

### TANGENT PLANES TO DOUBLE CURVED SURFACES.

492. Art. 378 presents the principles on which problems under this head are solved. Refer to cases (*e*) and (*f*) of Art. 446, if necessary, as to the projections of points on double curved surfaces.

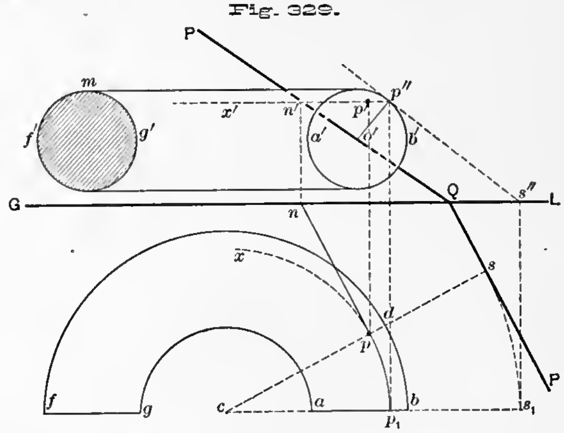
493. *A plane, tangent to a sphere at a given point*, is perpendicular to the radius drawn to that point; hence in Fig. 328 we make a plane $P'QP$ tangent to the sphere at $o\,o'$ by drawing the contact-radius $o\,c$, $o'c'$, and then, by Art. 317, passing a plane perpendicular to it at its extremity, $o\,o'$; $o\,d$, $o'd'$—a "horizontal" (Art. 300) of the plane sought—giving $d'$, through which $P'Q$ is drawn perpendicular to $o'c'$.

Fig. 328.




494. *A plane, tangent to an annular torus at a given point*, would be determined by the tangents to the circular sections containing the point; or as in the last problem, by being made perpendicular to the radius of the circular section in the meridian plane through the point. Illustrating only the first method, we take through the point $p\,p'$, Fig. 329, a *meridian section* $c\,d$ (elevation unnecessary), and a *parallel*, $p_1\,p\,x$. Carrying the former about the axis $c$ until parallel to V it appears at $a'p''b'$, $a\,b$, and $p'$ reaches $p''$, at which the tangent $p''s''$ is drawn. The trace $s_1$ of the tangent counter-revolves to $s$, through which the h.t. of the desired plane ($P\,s\,Q$) is drawn, parallel to $n\,p$, which is not only a tangent at $p$ to the parallel $p_1\,p\,x$, but also a horizontal of the plane sought.

Fig. 329.




The v.t. of the plane joins $Q$ with $n'$, the v.t. of the horizontal $n\,p$.

495. *A plane, tangent to a sphere and containing a given line*, may be found on this principle: A plane through the centre of the sphere and perpendicular to the line would cut the *sphere* in a great circle, the *line* in a point; either tangent that could be drawn from the point to the circle would—with the given line—determine a plane fulfilling the conditions.

If the given line is *tangent* to the sphere there is but one solution, and if it intersects it the problem is impossible.

In Fig. 330 let $a\,b$, $a'b'$ be the given line, and $o\,o'$ the centre of the given sphere. The plane $e\,d'c'$ is then drawn, containing $o\,o'$ and perpendicular to the given line, determined by means of the horizontal $o\,c$, $o'c'$. (Art. 300).

The line and plane intersect at $s\,s'$. (Art. 322). After rabatment into H about $e\,d'$ as an axis we find $s$ at $s_1$, and $o$ at $o_1$, the latter the centre of the revolved great circle cut from the sphere by plane $e\,d'n'$. (Art. 306).

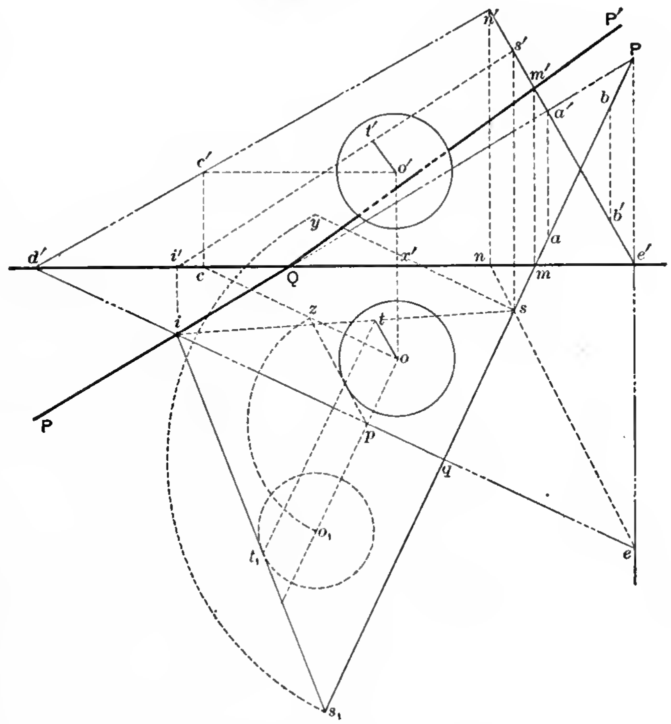
Draw the tangent $s_1\,t_1$. It meets $e\,d'$ at $i$, which is not only the h.t. of the tangent, and as such may be joined with the like trace of $a\,b$, $a'b'$ to give $P\,Q$, but being also a constant point during rotation may be joined with $s$ to give the plan of the tangent when in its true position. Then $t_1$ projects upon $i\,s$ at $t$, whence $o\,t$ as the contact radius.

Projecting $i$ to $i'$ and joining with $s'$ we have the v.p. of the tangent, upon which $t$ projects at

$t'$, whence $o't'$ follows for the v.p. of $ot$. $P'Q$ passes through $m'$, the v.t. of the given line, and is perpendicular to $o't'$, either of which condition is, with $Q$, sufficient to determine it.

496. *A plane, containing a given line and tangent to a given sphere,* may also be found *by means of an auxiliary cone,* thus: Make any point of the given line the vertex of a right cone which is tangent to the sphere; then either tangent to their circle of contact, drawn from the trace of the given line upon the plane of that circle, will — with the given line — determine a plane meeting the requirements.

Fig. 330.



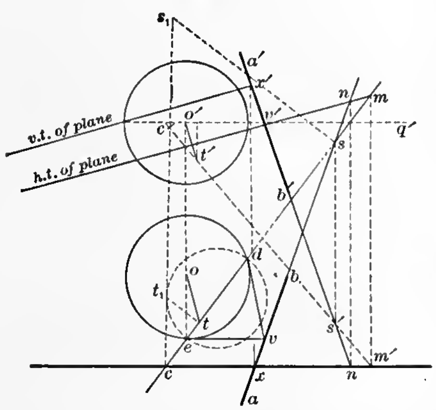
Should the given line happen to be *parallel* to the plane of the circle of contact of cone and sphere, the required plane would be determined by (a) the given line, and (b) a line parallel to the given line and tangent to the circle of contact.

The axis of the auxiliary cone may preferably be *parallel* to either H or V, so that the base may be projected as a straight line on that plane.

In Fig. 331, deciding that the axis of the auxiliary cone shall be horizontal, we take for its vertex $v v'$ that point of the given line $a b$, $a' b'$, that is *at the same level* as the centre $o o'$ of the sphere; then $v e d$ is the plane of the tangent cone, and $e d$ that of its circle of contact.

The plane of $e d$ meets the given line at $s s'$. Using for an axis the line $e d$ (regarded now as the diameter of the circle of contact) we find $s$ and said circle appearing at $s_1$ and $d t_1 e$, when brought into a horizontal plane.

Fig. 331.




Draw the tangent $s_1 c$ and project $c$ upon $q'o'$ at $c'$; then in counter-revolution the tangent *line* becomes $s c$, $s'c'$, and the tangent *point* $t$, $t'$, whence $o t$, $o't'$ follows for the contact radius.

Through the traces ($x'$ and $n$) of the given line the traces of the required plane are last drawn, perpendicular to $o't'$ and $o t$ respectively.

497. *A third method of determining a plane through a line and tangent to a sphere* is to envelope the sphere by two tangent cones whose vertices are on the given line. The common chord of their circles of contact will pierce the surface of the sphere at points, either of which will — with the given line — determine a plane fulfilling the conditions.

498. *A tangent plane to any surface of revolution, at a given point P,* is perpendicular to the meridian (axial) plane containing the point. For a tangent at $P$ to a *parallel* of the surface would be perpendicular to its radius, and also to a line drawn through $P$ parallel to the axis; hence any plane through such tangent would be perpendicular to the plane that would be determined

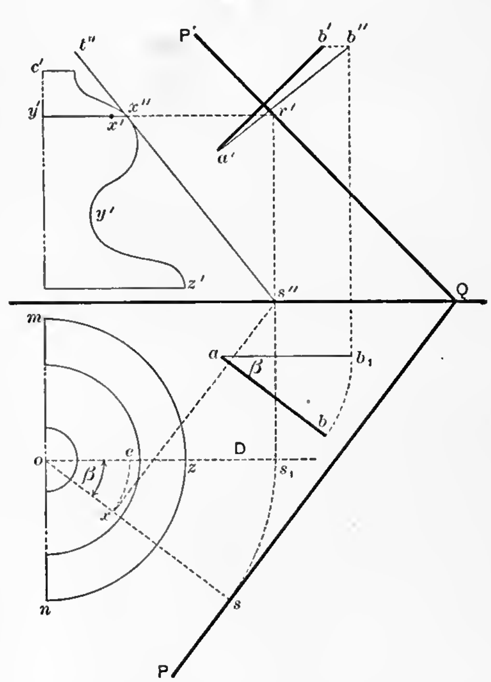
by such radius and axis-parallel, which is evidently none other than the meridian plane.

499. If two con-axial surfaces of revolution are not tangent to each other but have a common tangent plane, the points of contact of the latter with them will, by Art. 487, lie in the same meridian plane, and the line joining them will not only be a tangent to both meridian curves in that plane, but will also be one of the determining lines of the common tangent plane.

500. *A plane containing a given line and tangent to any surface of revolution, obtained by means of an auxiliary con-axial surface.*

With the given line generate an auxiliary hyperboloid, con-axial with the given surface. In *any* meridian plane draw the common tangent of the meridian curves of the original and auxiliary surfaces; it will, by the last article, be one of the determining lines of a plane, tangent to the surfaces at the contact-points on the curves. Rotate this tangent about the axis until it intersects the original line, when, with the latter, it will determine the plane sought.

Fig. 332.




501. *A plane containing an exterior point and tangent to a double-curved surface of revolution,* on either a *parallel* or a *meridian.*

In the first case the plane will be tangent to a cone having the parallel for its base and the point for its vertex.

In the second case the determining lines will be (a) the perpendicular from the point to the plane of the given meridian, and (b) the tangent to the meridian curve from the foot of such perpendicular.

502. *A plane, perpendicular to a given line and tangent to a double curved surface of revolution,* may be found thus: In Fig. 332, if $a\,b$, $a'\,b'$ is the given line, rotate it till parallel to V, when it will become $a\,b_1$, $a'\,b''$; then $t''x''s''$ drawn perpendicular to $a'\,b''$ and tangent to the meridian section $x''y'z'$, will represent—in revolved position—one of the planes sought; and, after counter-revolution through an angle $\beta$ equal to that between $a\,b$ and $a\,b_1$, we find $x''(e)$ at $x',x$, the point of tangency, and $P\,Q\,P'$ as one of the desired planes, found as in Art. 317.

INTERSECTING SURFACES.

503. As illustrating, among other things, the serviceability of the *ground line*, which has not appeared—as such—in the problems of intersection treated by the Third Angle Method earlier in this chapter, a few cases of interpenetration are next solved in the First Angle.

504. The general principles on which an outline of intersection is obtained may be reviewed in Arts. 421–424. Those relating to tangents are repeated in the next articles.

505. The *tangent* at any point of a *plane* curve of intersection will be *the line of intersection* of the plane of the curve with a plane that is tangent to the surface at the given point.

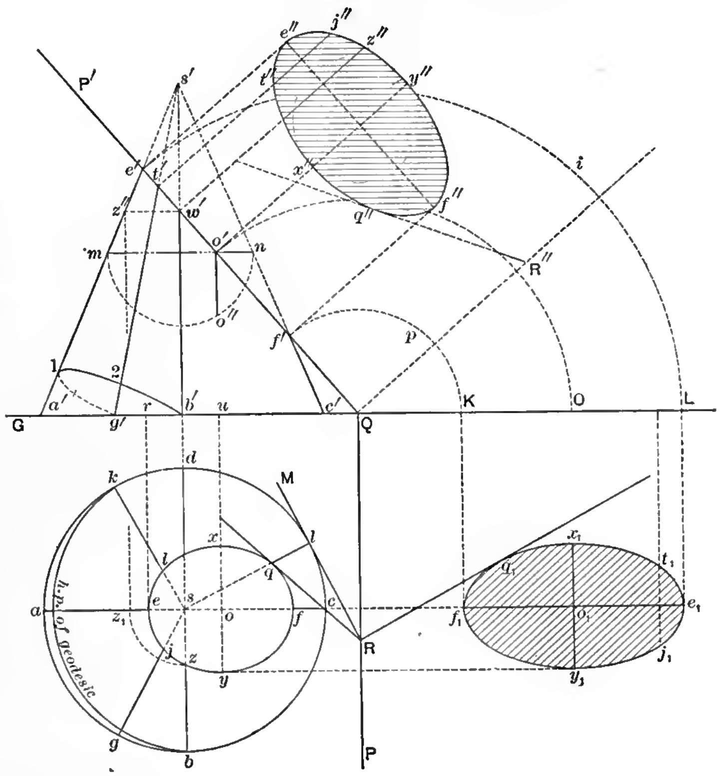
506. The *tangent* at any point of a *non-plane* curve of intersection is *the line of intersection* of two planes, each of which is tangent, at the given point, to one of the surfaces.

507. *The intersection of a vertical right cone by a plane.* In Fig. 333 $s'.a'c'$, $s.\,a\,b\,c$, is a vertical right cone, of altitude $s'\,b'$. $P\,Q\,P'$ is a plane of section, oblique to the horizontal but perpendicular

to V. It cuts from the cone an ellipse seen in $e'f'$, which is also the v.p. of the major axis.

Bisect $e'f'$ at $o'$. Through $o'$ take a right section $mn$. Rotate half the latter till parallel to V, when it will appear as $mo''n$. Then $o'o''$ is the semi-diameter of the ellipse sought, which can be constructed by any of the methods involving simply the knowledge of the axes; or, as we shall see later, by employing the plan $eyfx$, whose construction is next described.

Fig. 333.




*The horizontal projection of the curve of intersection.* Since the plane $PQP'$ is at 90° to V we can project directly from $e'$, $t'$, etc., where the trace $P'Q$ crosses the elevations of elements, to the plans of the latter. Thus $e'$ and $f'$ project at $e$ and $f$, upon the plans of the extreme elements. On $sk$ and $sg$, whose common elevation is $s'g'$, we get $t$ and $j$ from $t'$.

To get $z$, on the foremost element $sb$, $s'b'$, project $w'$ to $z''$, thence to $z_1$ and by arc (centre $s$) to $z$.

Project $o'$ upon $ef$ at $o$, through which $xy$ (equal to $2o'o''$) at 90° to $ef$ will be the h.p. of the minor axis, seen in true size because horizontal.

*True size of the section, found by revolution into the horizontal plane.* At $f_1y_1e_1x_1$ the true ellipse is shown by revolution of its plane into H, about $PQ$ as an axis. The arcs described by the various points will be projected upon V in their true size, all with centre $Q$: thus, $e'$ describes $e'iL$; similarly, $o'$ ($x$, $y$) reaches $O$; then $L$ and $O$ project at $e_1$, $x_1$, and $y_1$, upon perpendiculars to $PQ$ as $ee_1$ and $yy_1$, which are the plans of the arcs of rotation.

*True size of the section, shown by revolution into the vertical plane.* Through the v.p. of any point revolved, as $e'$, draw a line at 90° to $P'Q$ to represent the v.p. of the arc of rotation, and on it lay off $e'e''$ equal to the distance of the point in space from the axis $P'Q$, which distance, being horizontal in this case, is shown by the distance $er$ of its plan from G.L. Similarly, $o'x''$ equals $ux$; $o'y''$ equals $uy$.

*The tangent line at any point of the curve.* Let a tangent be desired at some point $q$. $MR$ is the h.t. of a plane that is tangent to the cone along the element $st$ on which $q$ lies, and $R$ is a point of the intersection of the tangent plane and section plane; $Rq$ is therefore the line of intersection of those planes, hence the tangent line required. (Art. 505).

When $q$ reaches $q_1$ the tangent is $Rq_1$, $R$ being constant during the rotation, being on the axis. Make $QR''$ equal to $QR$; then $R''q''$ is the tangent, rotated into V.

*The development of the cone.* With radius $SC$ (Fig. 334) equal to the slant height of the cone, draw an arc $CAC$, subtending an angle of 144°, determining the latter by calculation in the proportion $R\ (a's') : r\ (as) :: 360 : \theta$; since in *unequal* circles *equal* arcs are subtended by angles at the centre which are inversely proportional to the radii.

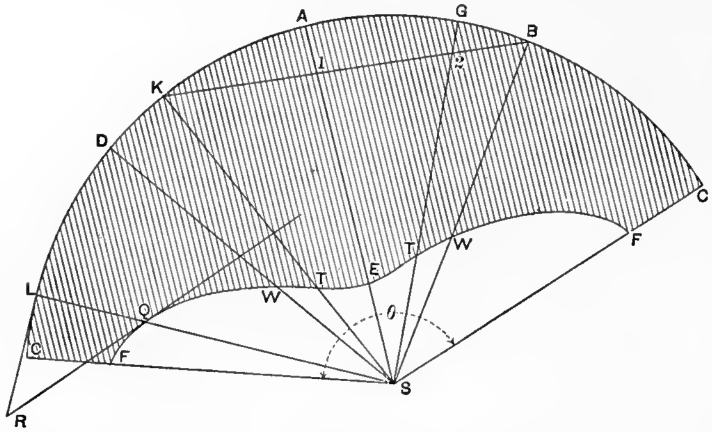
Locating the elements $SL$, $SD$, etc., on the sector $CSC$, lay off on each the distance from $S$ of

the point where that element is cut by the plane in Fig. 333. Thus, $SE = s'e'$; $SW = s'z''$, the true length of $s'w'$.

*The tangent line in development* appears at $LR$, drawn at 90° to $SL$ and made equal to $lR$ in Fig. 333.

508. *The shortest distance between two points on the surface of a cone*, i.e., their *geodesic*. (Art. 382).

Fig. 334.



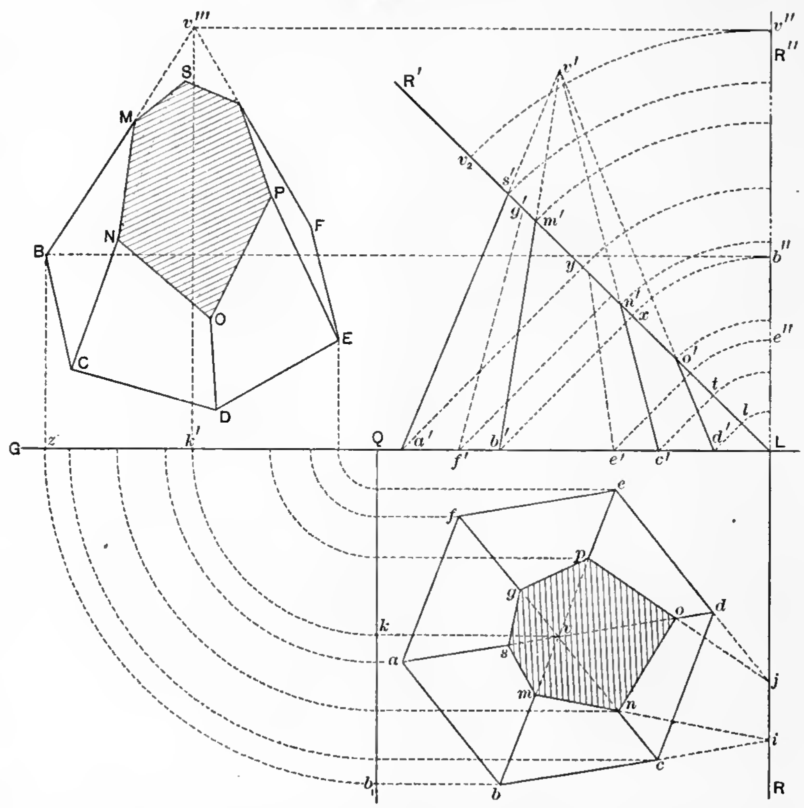

The shortest distance between two points would be a straight line on the development. Hence in Fig. 334 $BK$ will be the *geodesic* between $K$ and $B$. It crosses $SA$ at 1 and $SG$ at 2. Locate 1 and 2 on the same elements as seen on the cone in Fig. 333; then the curve $g'1b'$ is the elevation of the geodesic.

*Conical helix.* In Art. 191 the curve traced by a point which combines a uniform approach toward the vertex of a cone with a uniform rotation about its axis is called a *conical helix*, from the analogy of its generation to that of the cylindrical helix. But if we follow Javary and base the definition upon the geodesic property of the locus, the curve we have just constructed is also entitled to the same name, although there is evidently nothing in its form to suggest what is usually understood by a helical curve.

509. *Intersection of a pyramid by a plane; also sectional view.* In Fig. 335 let $v.ab..f$, $v.'a'd'$ be a vertical pyramid, cut by a plane $R'LR$ that is perpendicular to the vertical plane.

Project $s'$, where $R'L$ crosses $v'a'$ (the elevation of an edge), down to $s$ upon $va$, the h.p. of the same edge. Similarly, get $m$ from $m'$, $n$ from $n'$, and complete the shaded plan of the section.

Fig. 335.



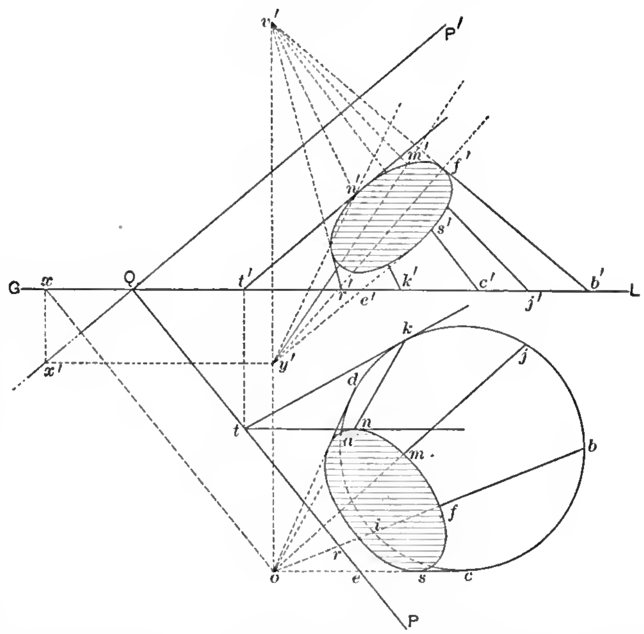

*Another way* of getting all points of the section *but one*, is to use the intersections of the trace $RL$ with the H-traces of the various faces. Thus, by the first method, get $o$ from $o'$, with which to start; then, as $ed$ is the h.t. of the face $ved$, we shall have $j$ as one point of the intersection of that face with the given plane, and $jo$ for the line itself, $op$ being that portion of it which lies within the limits of the face considered. In like

manner, $p\,g$, $f\,e$ and $R\,L$ would all meet in one point, and, correspondingly, all analogous sets of three.

*For the sectional view* $v''_1 B\,D\,E$ project all points upon $R'L\,R$ by perpendiculars, as $b'x$; rotate $R'L\,R$ upon $R\,L$ till vertical, at $R''L\,R$; transfer it to $Q\,b_1$ and finally rotate it into V on the left.

510. *Plane sections of pyramids, cones, etc., are homologous with the bases of the surfaces cut; and the intersection of the cutting plane with the plane of the base is an axis of homology.* (Art. 145).

Fig. 336.

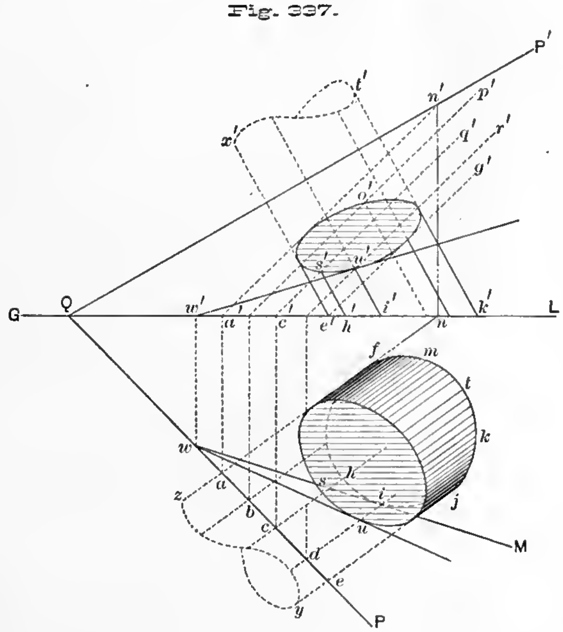


In Fig. 335, if $v\,v'$ is a *centre of projection*, then $c$ is the projection of $p$; $d$ of $o$, etc. Also, $p\,o$ and $e\,d$ meet at $j$ on the trace $R\,L$; $m\,n$ and $b\,c$ meet at $i$; and similarly for the other lines and their projections. $R\,L$ is, therefore, an *axis of homology*.

511. *The intersection of an oblique cone by a plane oblique to both* H *and* V; *also, a tangent to the section.* In Fig. 336 let $v'.r'b'$, $o.c\,b\,k$, be the cone; $P\,Q\,P'$ the plane. Let fall a vertical line through the vertex of the pyramid. Its plan will be $o$, while $v'y'$ will be part of its elevation, and $y'$ is the elevation of its intersection with the plane $P\,Q\,P'$, found by Art. 322.

In $o\,d$, $o\,k$, $o\,j$, etc., we have the traces of auxiliary *vertical* planes through the vertex. These planes must cut $P\,Q\,P'$ in lines passing through $y'$. Hence, for any one, as $o\,b$, note $r$—its intersection with the trace $P\,Q$, and project to G. L. at $r'$, which join with $y'$; then $y'r'$ is the v. p. of the intersection of plane $P\,Q\,P'$ with plane $o\,b$, and at $f'$ meets $v'b'$, the highest element cut by that auxiliary. The point on the element $o\,i$ in the same plane is similarly found. The plans of the points $f'$, $m'$, etc., are then derived from the elevations.

Fig. 337.



*The tangent line*, at any point $n\,n'$ of the curve, is thus found: Draw the element $o\,n\,k$ through the point. Make $t\,k$ tangent to the base at $k$, for the h. t. of a tangent plane. It meets $P\,Q$ at $t$, one point of the intersection of the section and tangent planes. Then $t\,n$ is such intersection, and therefore the tangent sought. (Art. 505). The elevation of the tangent is then $t'n'$.

512. *The intersection of an oblique cylinder by a plane oblique to both* H *and* V; *also, a tangent to the curve.* In Fig. 337 let $P'Q\,P$ be the section plane. As auxiliary planes parallel to the axis of the cylinder will be the most convenient, take $z\,n\,n'$ as one such, vertical, and cutting $P'Q\,P$ in the line $a'n'$. Then all planes parallel to $z\,n\,n$—as $y\,j$, $c\,l$—will intersect $P\,Q\,P'$ in lines $c'g'$, $e'q'$, etc., which will be parallel to $a'n'$. Any auxiliary plane $c\,l$ cuts elements from the cylinder which will meet $c'q'$ in points of the curve, as $s'$, from which $s$ is obtained.

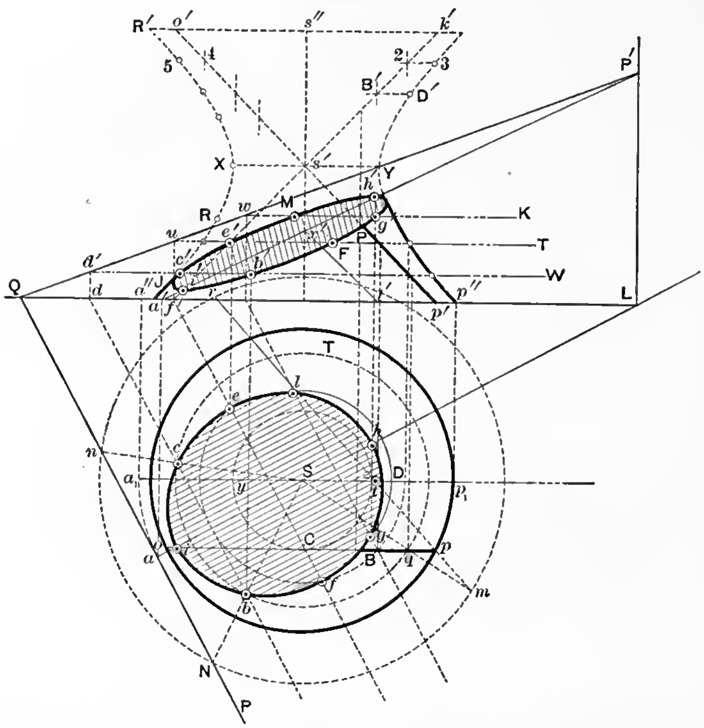
*The tangent line* is drawn, as for the cone, by making $Mw$ tangent to the base at $i$—the extremity of the element through $u$, the point of tangency—and joining $w$ with $u$; also $w'$ with $u'$. (Art. 505).

513. *The intersection of an hyperboloid of revolution by a plane that is oblique to both* H *and* V.

This problem may be taken as illustrating also the general problem of the intersection of a plane with a surface of revolution whose axis is perpendicular to a plane of projection.

Let $P'QP$ (Fig. 338) be the cutting plane, and let the hyperboloid be found as follows: Having given the altitude $= 2s's''$, the gorge diameter $= XY$, and $45°$ for the inclination of the elements, draw circle $yCz$ for the plan of the gorge; $p's'o'$ through $s'$ and at $45°$ to $QL$; project $p'$ and $o'$ at $p$ and $o$ upon a line parallel to $QL$ and tangent to $yCz$. Then the circle of radius $Sp$ is the base of the surface. (Art. 468).

Fig. 338.




Since the surface is doubly ruled draw $f'k'$ through $s'$ and at $45°$, for the other element having $op$ for its plan.

Take any plan $B$ on $op$, and project up to both $o'p'$ and $f'k'$. Carry $B$ by arc with centre $S$ to $D$, in the meridian plane $aS$ parallel to V; then project $D$ to the *level* of the original elevations, getting $D'$ and a point on $TF$, not lettered but symmetrical with $D'$ as to the axis $XY$ of the hyperbola. Similarly for other points in the hyperbolic contour or meridian section.

*The section.* Auxiliary *horizontal planes* will cut *circles* from the hyperboloid, and *straight lines* from the plane $PQP'$.

$Wd'$ is one such horizontal plane. It cuts the circle $cTq$ from the hyperboloid, and the line $db$ from $PQP'$. These meet at $c$ and $b$, two points of the section sought. These plans project up to the plane $Wd'$, giving $c'$ and $b'$. An analogous process with each of the other auxiliaries, and with some which are omitted from the figure, gives the elliptical section shown.

*The highest and lowest points,* $h'$ and $i'$, lie in that meridian plane $aLP'$ which is perpendicular to $P'QP$. To find $i'$ carry the line of declivity $Sa$ $(a'x')$ to $Sa_1$, when—being parallel to V—its elevation $a''x'$ will be parallel to $P'Q$, $x'$ being the intersection of $PQP'$ with the axis. Then where $a''x'$ crosses the hyperbola $JXR'$ will give the *level* of the desired point $i'$ on $a'x'$. Similarly, the crossing of $a''x'$ with $D'Yp''$ gives the *level* of $h$.

514. *Plane section of an hyperboloid of revolution, found by means of auxiliary cones.* As illustrating the use of other auxiliaries than the customary *planes*, a few lines on Fig. 338 show how the same points could have been otherwise obtained. Use $x'$—the intersection of plane $PQP'$ with the axis $s's''$—as the vertex of a series of vertical *cones*. Each of these will, in general, cut the hyperboloid in two circles, and the plane $PQP'$ in two right lines. The intersection of these circles and lines will be points of the curve.

Let $nNm$ be the base of one auxiliary cone. It cuts the plane $PQP'$ in the lines $ns$ and $NS$. One point of each circle in which the cone cuts the hyperboloid will be found on the element

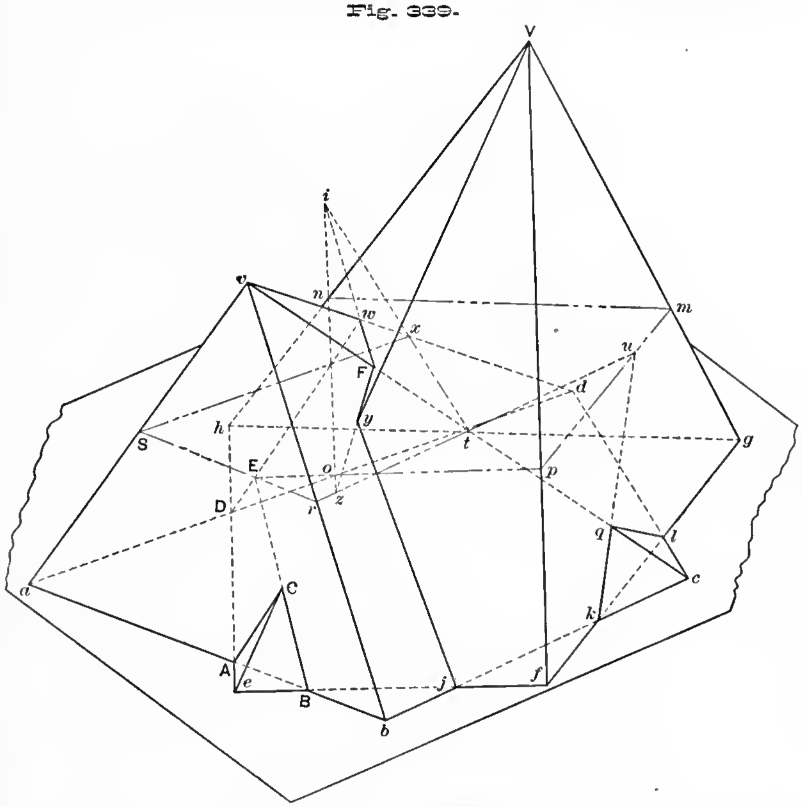
$op$. To find either, pass a plane through $op$, $o'p'$ and the vertex $x'(S)$. Its h. t. is $mr$, passing through $p$—the h. t. of the element, and through $t$, the h. t. of $x't'$, $St$, which is a line drawn through the vertex and parallel to the element. Then $Sm$ is the element cut from the cone by the plane containing $op$, and $q$ therefore a point of the circular intersection of the cone and hyperboloid. The circle of radius $Sq$ then gives $b$ and $c$ on the elements lying in both cone and plane.

Joining $S$ with $r$—the second intersection of $mpl$ with the base—would give a second element on the cone. It would meet $op$ at a point which would be used like $q$ to obtain two more points on the elements $nS$ and $NS$.

515. *The intersection of surfaces with bases in one plane, found by means of one auxiliary plane.*

Taking two pyramids (Fig. 339) to illustrate the general problem, we note first the points $A$, $B$, $j$, $k$, etc., in which the *bases* of the two surfaces meet, they being, evidently, points of the lines of intersection sought.

Fig. 339.




Any auxiliary plane *parallel to the bases* would cut sections of the same *form* as the bases. $Srdx$ is one such section, and $opmn$ another, determined as follows: Knowing that the altitudes of the pyramids are in this case as 2 to 1, any plane which *bisected* the edges and altitude of the smaller pyramid $v.abe$ would be *one-third* of the way from base to vertex on the pyramid $V.efgh$; join, therefore, the middle points of the edges of $v.abed$, while on the other take $fp$ one-third of $fV$, and similarly locate $m$, $n$ and $o$, for the second section. $Sr$, parallel to $ab$, meets $po$—the parallel to $ef$—at $E$; then $BE$ is the intersection of the planes of faces $Vef$ and $vab$, real, however, only to $C$, where the actual boundary of $Vef$ is reached and the intersection runs on to the face $Vhe$. But, having $A$, we have only to draw $CA$ to complete that part of the construction. Similarly, join the intersection of any two edges of the bases with the intersection of the corresponding edges in the auxiliary sections, using the line thus obtained only up to the point where it reaches the actual limit of either of the faces on which it lies.

516. *In orthographic projection a similar problem to the last* is worked in Fig. 340, in *plan* only, the drawing of an elevation being left for the student.

The altitudes of the pyramids, 5″ and 4″, are indicated at their vertices, $v$ and $w$.

The section $r_1s_1t_1$ bisects the edges of the $w$-pyramid, being taken 2″ above the base.

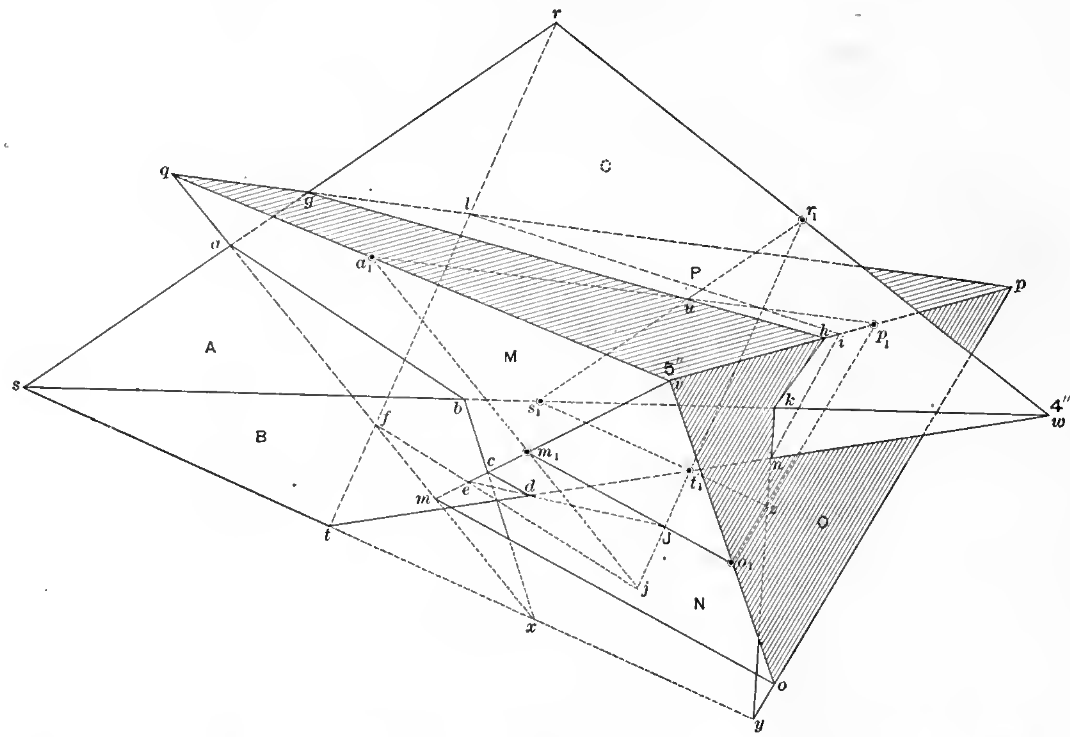
The section $a_1m_1o_1p_1$, *at the same level* as $r_1s_1t_1$, is therefore two-fifths of the way from the base $mopq$ to the vertex $v$. Starting at any point, as $f$, which is the intersection of the base lines of the faces $vqm$ and $wrt$, draw a line toward the intersection $(j)$ of $a_1m_1$ and $r_1t_1$, since these

are the parallels, at the upper level, to the base lines whose intersection with each other gave *f*.

At *e*, where *fj* runs off the face *vqm*, we turn towards *J*, one point of the intersection of face *vmo* with *wrt*, but can use *eJ* no farther than *d*, where the limit of face *wrt* is reached, and the intersection becomes that of faces *wts* and *vmo*. Continuing this process (for which nearly all the construction lines are given) results in the outlines *abcdef* on the one side, and *ghknil* on the other.

For the *developments* the student may refer, if necessary, to Art. 396 (Case 6), or to Art. 429.

Fig. 340.




517. *The intersection of two cylinders, two prisms, or of a cylinder and prism, when their bases lie in the same plane.*

Each plane, in a system of auxiliary planes passed parallel to the axes of the two surfaces, would cut their surfaces — if at all — in lines parallel to the axes, which would meet at points of the desired outline of intersection.

In Fig. 341, taking two *prisms* to illustrate the method just outlined, let *azxm* be the lower base of the one, and *kwdnf* that of the other.

As one way of determining the directions of the edges, let it be required that the edge starting from *m* shall meet the one beginning at *n* at some point $ss'$, whose position is assigned; then *msp* is the plan of one edge, and parallel to it will be the edges and axis of the prism to which it belongs. Similarly, *nsl* is a plan, and gives a direction for the other prism.

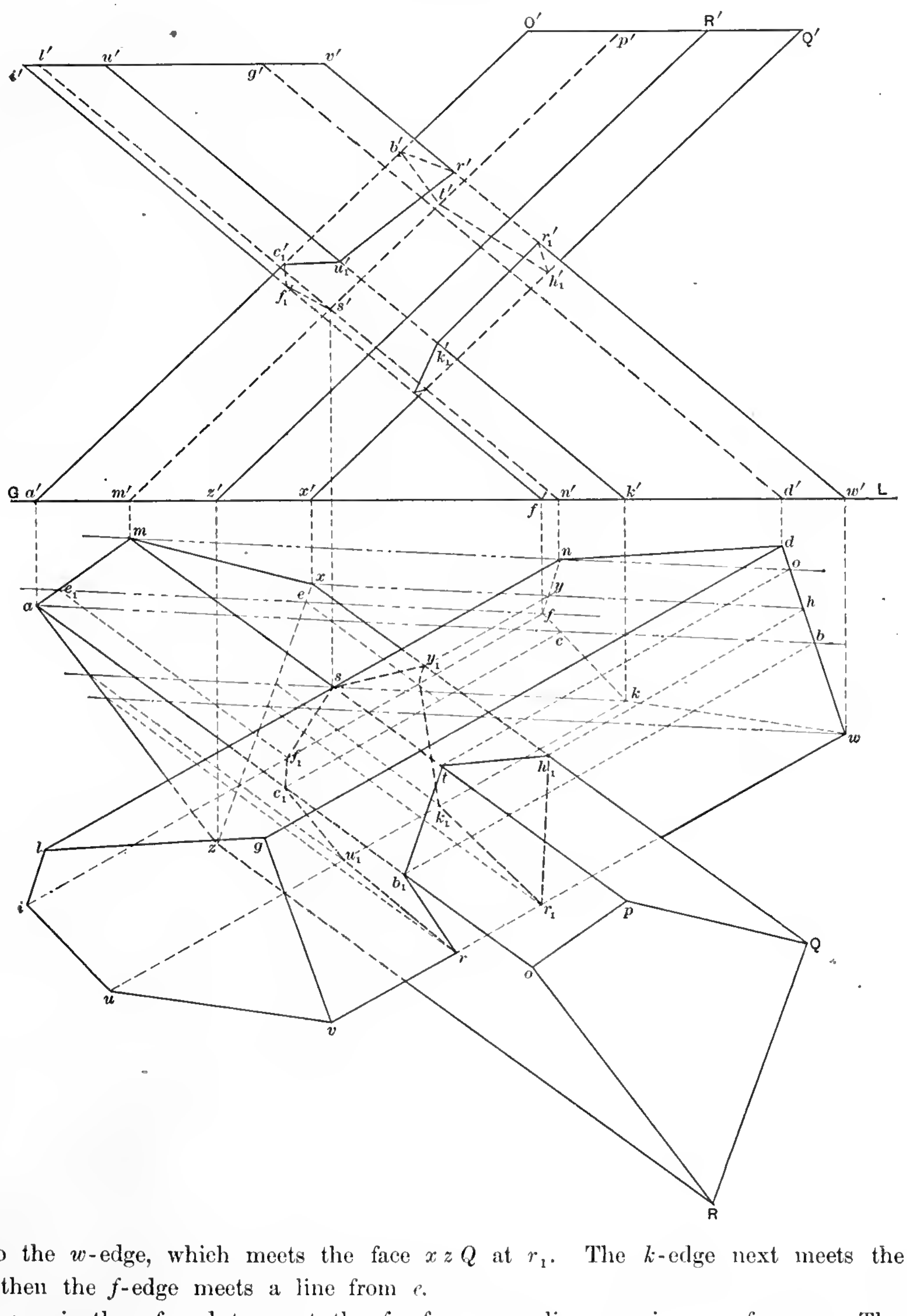
Project *m* to $m'$; draw $m's'p'$, and parallel thereto draw the other edges $a'O'$, $z'R'$, $x'Q'$. Then $n's'l'$ and its parallels complete the elevation of the other prism.

It having been predetermined that *mp* and *nl* should intersect, we have in *mn* the horizontal trace of the plane containing those two edges, and all other auxiliary planes parallel thereto will have H-traces parallel to *mn*. These are seen in *xh*, *ab*, etc.

The plane $m\,n\,o$ cuts the face $d\,w\,v\,g$ in a line $o\,t$ which meets the edge $m\,p$ in the point $t$.

Imagining the auxiliary plane advancing from $m\,n\,o$—its rear position, and, at first, noting intersections on the right-hand side of each surface, we find $x\,h$ as the next position in which it contains an *edge* of either surface; then the edge $x$ meets the line from $h$ at $h_1$. Further advance

Fig. 341.




brings us to the $w$-edge, which meets the face $x\,z\,Q$ at $r_1$. The $k$-edge next meets the same face at $k_1$, and then the $f$-edge meets a line from $c$.

The edge $x$ is then found to meet the $fn$ face on a line running up from $y$. The next move

completes the circuit of the right-hand base, with the exception of the vertex $d$, a plane through which would be entirely exterior to $mxza$, showing that the edge $dg$ clears the other prism.

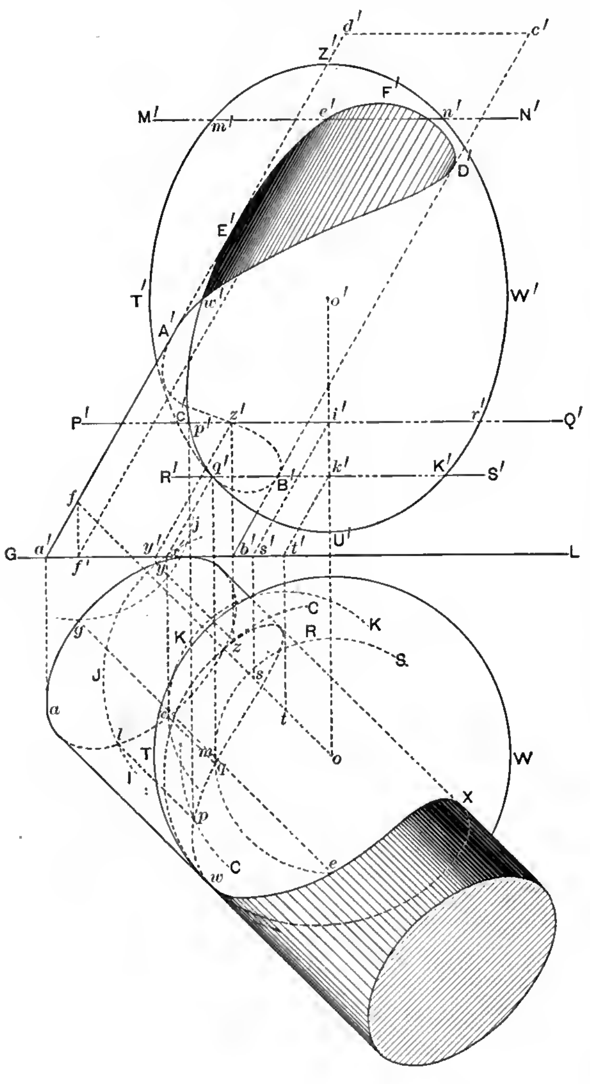
We next return from $n$ to $o$, while moving from $m$ to $z$ by the way of $a$; and the first plane to contain an edge is $fe$, resulting in $f_1$ on $fi$. Next $ac$ gives $c_1$; $ks$ gives $u_1$, while $r$ falls on the auxiliary plane through $w$. Plane $ab$ gives $b_1$, which joins with $t$ to complete the solution.

The elevation of the intersection is most readily obtained by projecting up to the edges the points that have just been determined in plan.

The visible portions of the surfaces are evidently not the same in the two views.

518. *Intersection of any surface of revolution by a cylindrical surface, by means of auxiliary cylinders.* In Fig. 342 we have the ellipsoid $T'U'W'Z'$ $(TW)$ as the surface of revolution. Let it be intersected by an oblique cylinder having an elliptical base, $agyzx$.

Fig. 342.




A series of random horizontal planes, $M'N'$, $P'Q'$, $R'S'$, cut circles from the ellipsoid, as $m'n'$, $c'r'$, each of which can be made the directrix of a cylindrical surface, whose elements will be parallel to those of the given cylinder. These auxiliary cylinders will intersect, if at all, in a common element, which meets the original plane at a point of the curve sought.

Taking, in particular, the plane $P'Q'$ with which to illustrate, we have the circle $C'r'$ projected at $CpC$ (centre $o$), and also at $IJy$, drawn from centre $s$, where $os$, $i's'$ is parallel to the elements of the cylinder. Then $lp$ and $yz$ are the elements cut from the original cylinder by the auxiliary cylinder having base $IJy$; and $p$ and $z$, their intersections with $CpC$, are points in the desired curve, and project in elevation upon $P'Q'$ at $p'$ and $z'$.

Similarly, $q'K'$ projects both to $SRm$ and to $KKx$ (centre $t$). Then $xq$ meets $SRm$ at $q$, which projects to $q'$ on the plane $R'S'$.

519. *The intersection of a surface of revolution with a conical surface, by means of auxiliary cones.*

Substituting auxiliary cones for the auxiliary cylinders of the last problem, the solution would be in strictest analogy to the one there described.

520. *The intersection of a sphere by an oblique cone whose vertex is at the centre of the sphere.*

In Fig. 343 a quarter sphere is shown in $s'M'R'$ and $MsN$. The cone is $s'.a'b'$, $s.apf$.

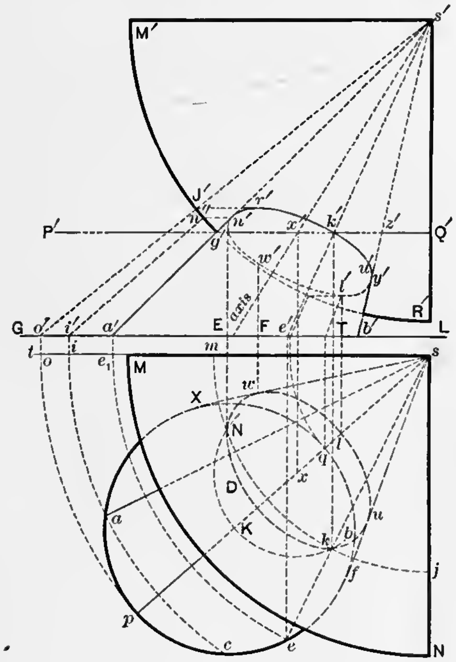
$P'Q'$ is an auxiliary horizontal plane. It cuts from the cone a circle of diameter $g'z'$, centre $x'$; seen in plan in $NKb$, centre $x$.

$Q'g'$ is the radius of the circle cut from the sphere by $P'Q'$, of which a quadrant is seen in plan at $mNk$. $N$ and $k$, the intersections of the auxiliary circles in the plane $P'Q'$ are then the

plans of two points of the curve sought. These are projected upon $P'Q'$ for the elevations.

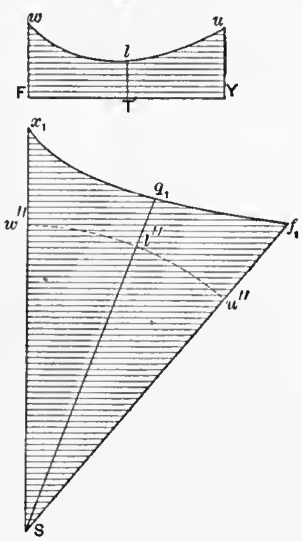
The highest and lowest points are those lying in the vertical meridian plane $sKp$. To find the former, carry $sp$ to $so$, parallel to V. Then $o's'$ is its new v. p. This cuts the spherical contour at $J'$, the *level* of $r'$, which lies on the elevation (not drawn) of the element $sp$. A similar procedure with element $sq$ gives $l'$, the lowest point.

Fig. 343.




521. *The development of a cone by means of a sphere whose centre coincides with the vertex of the cone.* Find the intersection, $n'k'y'$, of the cone and auxiliary sphere, as in Fig. 343. That part of the cone that is between the vertex and the intersection with the sphere will develop into a circular sector, since all points of the curve $n'k'y'$ are equidistant from the centre $s'$.

Fig. 344.




We have first to find the length of the curve of intersection, and then lay it off on a circle of radius $s'M'$. To do this, take the vertical projecting cylinder of the curve. Its plan is $Ndkl$. On developing this cylinder the curve in question will roll out in a new curve, a portion of which is seen in the upper part of Fig. 344, $FTY$ being the rectification of the arc $wlu$ in Fig. 343. Then, with $Fw$ (Fig. 344) made equal to $Fw'$ (Fig. 343), $Tl$ to $Tl'$, etc., we have the curve $wlu$, Fig. 344, for the cylindrical development of the arc $w'l'u'$.

Fig. 345.

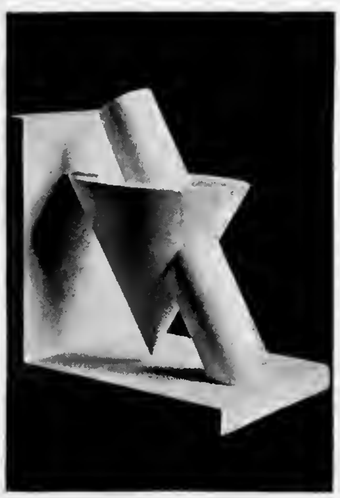


Fig. 346.

Draw next the arc $w''l''u''$, Fig. 344, of *radius* equal to that of the sphere, and of *length* equal to the $wlu$ above it; make the elements $Sx_1$, $Sq_1$, $Sf_1$, equal to the *true lengths* of the space elements as derived from their projections in Fig. 343. We then have in the figure $S.x_1q_1f_1$ the development of a portion of the given cone.

522. *Brush Shading of Intersecting Surfaces.* Problems in intersection are not only a valuable test of a student's acquaintance with the properties of surfaces and of his power to apply general principles to special cases, but also afford an especially advantageous field in which to exhibit skill with the brush, particularly if to the ability to represent form there be added a thorough acquaintance with the geometrical construction of shadows. As a rule, only elevations would be shaded.

In illustration of the foregoing Figs. 345 and 346 are introduced. Being reproductions of photographs of plaster casts, they show *natural* as distinguished from *conventional* light and shade effects.

# CHAPTER XI.

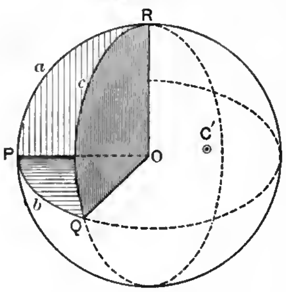
## TRIHEDRALS, OR THE SOLUTION OF SPHERICAL TRIANGLES BY PROJECTION.

523. The solid angle contained by three planes meeting at a given point is called a *trihedral angle.*

If the vertex of a trihedral is at the centre of a sphere, as in Fig. 347, the planes of the sides will cut the surface of the sphere in three arcs of great circles, forming the sides of what is called a *spherical triangle.*

The *arcs* (sides) are the measures of the plane angles whose common vertex is the centre of the sphere.

Fig. 347.




The *dihedral angle* $P\,O\,Q$ between the planes of two sides, as $a$ and $c$, is measured by whatever arc, $P\,Q$, is included by the planes upon the great circle having for its pole the intersection ($R$) of the sides. Since tangents at $R$ to the arcs $a$ and $c$ would be parallel to $O\,P$ and $O\,Q$ respectively, they would include the same angle as the planes in which they lie.

The solution of a spherical triangle consists in the determination of the dihedral angles between the planes of its sides, and the plane angles subtending the latter.

524. In Fig. 347 the sides $P\,R$, $P\,Q$ and $Q\,R$ of the spherical triangle $R\,P\,Q$ will be referred to as $a$, $b$ and $c$ respectively, the dihedral angle *opposite* each side being denoted by the same letter capitalized. Thus, the dihedral angle on edge $P\,O$ is called the angle $C$.

If $C'$, that pole of the arc $Q\,R$ which lies on the side of the plane opposite to the trihedral, be joined by arcs of great circles to the analogous poles of the other arcs, $a$ and $b$, we would have what is called a *polar* or *supplemental* triangle, $A'\,B'\,C'$, whose relations to the original or "primitive" triangle are shown by Spherical Geometry to be as follows:

(1) $A' = 180° - a$ . (4) $a' = 180° - A$.
(2) $B' = 180° - b$ . (5) $b' = 180° - B$.
(3) $C' = 180° - c$ . (6) $c' = 180° - C$.

For the primitive triangle these equations take this form:

(7) $A = 180° - a'$ . (10) $a = 180° - A'$
(8) $B = 180° - b'$ . (11) $b = 180° - B'$
(9) $C = 180° - c'$ . (12) $c = 180° - C'$

The three sides and three angles are called the *elements* of a spherical triangle, and with any three given the others may be determined.

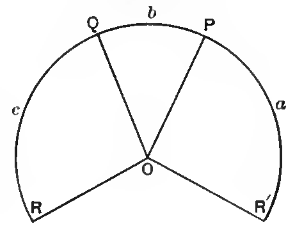
525. The six following cases may arise: We may have given (1) the three sides; (2) two sides and the included angle; (3) two sides and the angle opposite one of them; (4) the three angles; (5) two angles and the included side; (6) two angles and a side opposite one of them.

Although not often necessary, we may always reduce the last three cases to the form of the first three by means of the supplementary triangle. For example, Problem 4 may be worked by

solving Prob. 1, using sides which are the supplements of the angles given; then the supplements of the angles thus determined will be the sides desired.

The student will find it to his advantage to make cardboard models illustrating various cases. This can readily be done by cutting sectors of different sizes, and folding them on two of their radii. Thus, $O.R\,Q\,P\,R'$ (Fig. 348) is the sector which, folded on $O\,P$ and $O\,Q$, would illustrate the triangle of Fig. 347.

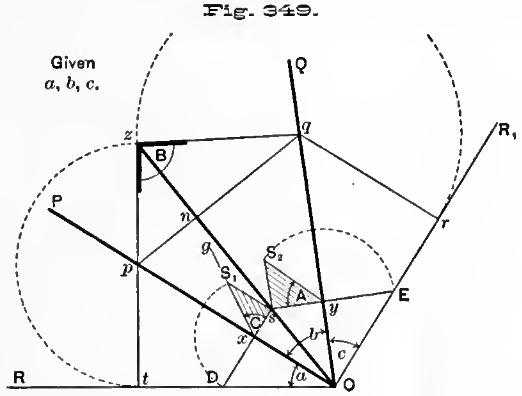
Fig. 348.



526. The following properties of the spherical triangle must be kept in mind: (1) The greater side lies opposite the greater angle, and conversely; (2) the sum of any two sides must be greater than the third; (3) the sum of the sides must be less than four right angles; (4) the sum of the angles must be greater than two right angles and less than six; (5) each angle must be less than 180°.

527. *To solve a spherical triangle having given the three sides.* (1) Take $a = 32°$; $b = 50°$; $c = 42°$. In Fig. 349 lay out these angles at $O$, obtaining the development of the trihedral on the plane of the side $b$. If we now rotate the faces $a$ and $c$ upon $O\,P$ and $O\,Q$ respectively, until $O\,R$ and $O\,R_1$ coincide, the trihedral will take its space-form.

Fig. 349.



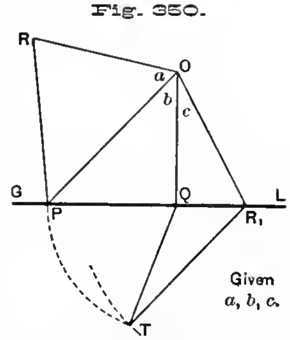

Make $O\,D = O\,E$; then, after the rotation supposed, the points $D$ and $E$ will coincide, each having turned in a vertical plane perpendicular to its axis of rotation. $D\,s$ and $E\,s$ are the traces of these planes of rotation, and $s$ is the plan of the united $D$ and $E$; $O\,s$ is therefore the plan of the united $O\,R$ and $O\,R_1$, that is, of the space-position of the third edge of the trihedral.

At $s$ draw perpendiculars to $s\,D$ and $s\,E$. Cut the former at $S_1$ by an arc of radius $x\,D$, centre $x$, and the latter at $S_2$ by an arc from centre $y$, radius $E\,y$. Then $s\,S_1$ obviously equals $s\,S_2$, while $s\,x\,S_1$ ($C$) and $s\,y\,S_2$ ($A$) are two of the three angles sought.

The plane of the third angle, $B$, being perpendicular to the edge $O\,s$ just determined, draw $p\,q$ at 90° to $O\,s$ to represent its trace on the plane of the side $b$. This plane will cut the faces $a$ and $c$, when in their space-positions, in lines perpendicular to their common edge, and seen in development at $p\,t$ and $q\,r$. Arcs $t\,z$ and $r\,z$, from centres $p$ and $q$, intersect at $z$, giving $p\,z\,q$ as the angle $B$ sought.

(2) *Solving upon both* H *and* V we may lay out in the latter (Fig. 350) the three faces $a$, $b$, $c$, with edge $O\,Q$ perpendicular to G. L. Make $O\,R = O\,R_1$; join $P$ with $R$; draw arc $P\,T$ from centre $Q$ and cut it from centre $R_1$ by an arc of radius $T\,R_1 = P\,R$. Then $O\,Q\,T$ is obviously the position taken by face $b$ when the latter has carried with it the face $a$ far enough for it to reach across to $O\,R_1$.

Fig. 350.



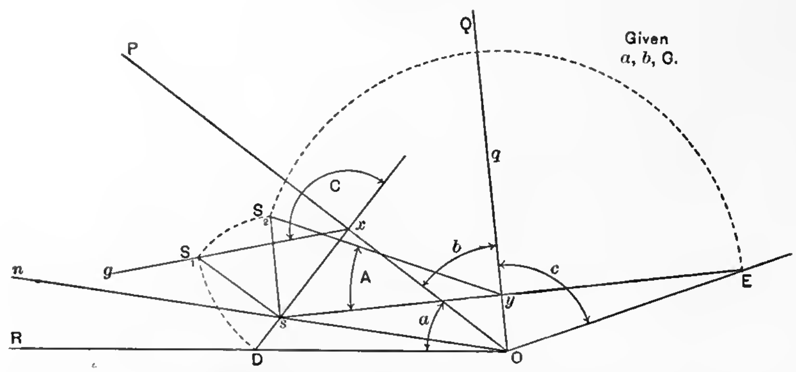

The angles between the faces may then be found by applying the usual solution for the angle between two intersecting planes.

528. *Given two sides and the included angle.* (1) Taking the same numerical values as in the last problem Fig. 351 illustrates the solution when the included angle is obtuse, while Fig. 352 is drawn for the case when said angle is acute.

Starting with the given sides, $a$ and $b$, in the horizontal plane, take some point $D$ on $O\,R$ and

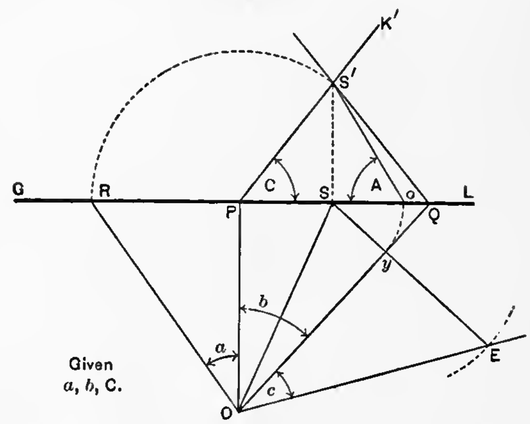
through it draw an indefinite perpendicular to $OP$, to represent the path of rotation of $D$. At $x$ construct an angle equal to the value given for $C$, and cut off its side $xg$ at $S_1$ by an arc of radius $xD$ from centre $x$. Then $S_1 s$, at 90° to $Dx$, gives $s$, through which $On$ is drawn for the plan of the edge $OR$ when in space-position.

Fig. 351.




Next draw $Sy$ perpendicular to $Oq$, for the trace of the plane of rotation of the point $S$ when regarded as in the side $c$. Draw $S_2 s$, at 90° to $sE$ and equal to $S_1 s$; then arc $S_2 E$, centre $y$, gives $E$, through which $OE$ follows for the third edge. The remainder of the solution is then in strictest analogy to that of the preceding problem.

Fig. 352.




(2) *Using two planes of projection* we would lay out the given faces $a$ and $b$ in either (in H, in Fig. 352). For convenience let the ground line be perpendicular to one of the edges, as $OP$. Lay out the angle $K'PO$ equal to the given angle $C$. Next rotate the face $a$ about $OP$ as an axis until $R$ falls on $PK'$, when its then position $S'$ will also be a point of the third face $c$; while $S'P$, $SO$, will be the projections of the space-edge.

$S'QO$ is the plane of the third side $c$. To find its inclination ($A$) to the face $b$ draw $Sy$ perpendicular to $OQ$, for the plan of a line of declivity of the plane. Carry $y$ into V at $o$ and there connect with $S'$, when in $S'oS$ we find in its true size the angle sought.

Make $yE = S'o$ and draw $OE$ for the edge of $c$, after development of the latter into H.

$OS$, being the intersection of $a$ and $c$, may be employed as in Art. 323, to find the angle $B$.

529. *Given, two sides and the angle opposite one of them.* From the fact that there are, usually, two solutions possible, this is called the ambiguous case.

(1) Illustrating first by a solution on two planes of projection (Fig. 353) take the given sides, $a$ and $b$, in H, with the ground line, $RQ$, at 90° to their common edge $OP$.

Let the angle $A$ be the other given element of the problem. Draw $qy$, at 90° to $OQ$, as the h.p. of a line of declivity of the face $c$; carry $y$ to $o$ from centre $q$, and make $qor = A$; $ro$ meeting the vertical $qr$ at a point of the v.t. of the face $c$. Draw $Qr$. Rotate face $a$ upon $OP$. The arc described by $R$ cuts $Qr$ at $S''$ and $S'$, either of which, connected with $O$, would give the space-position of the edge common to faces $a$ and $c$. If the former be used, draw $se$ perpendicular to $OQ$ and make $eE_2 = S''e''$, the true length of the line of declivity $se$. Then $QOE_2$ is the face $c$ for one solution, while $QOE_1$, analogously found, is its value for the $S'$-solution.

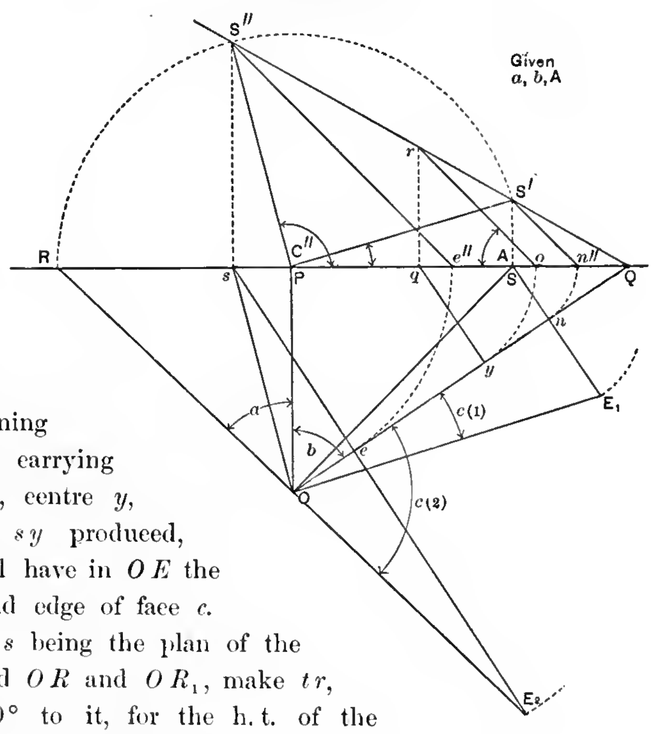
The angle $C$ is either $S''PQ$ or $S'PQ$, while $B$ is found as in Art. 323.

(2) *Using only the horizontal plane* in the solution, (Fig. 354) lay out $a$ and $b$ upon H, and anywhere on the face $b$ draw a line $og$ at 90° to $OQ$, for the h.t. of a vertical plane containing the angle $A$. Make $ogi$ equal to said angle, terminating $gi$ by a perpendicular to $og$ at $o$.

Draw $jQ$ at 90° to $OP$, as the h.t. of a plane in which some point $j$ of face $a$ would rotate.

Rotate such plane into H about $jQ$ as an axis. As $i$ was a point vertically above $o$ and *in* the face $c$, its new position $k$ is one point of the trace of $c$ upon the plane $jQ$; hence $QkF$ represents $c$ upon the rotation-plane of $j$.

Fig. 353.

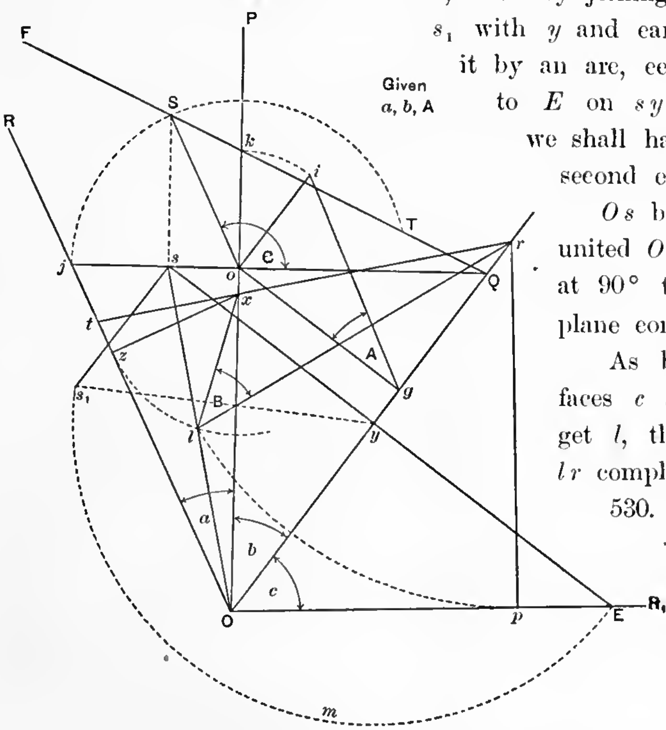



The arc $jST$, described by $j$, cuts $QF$ at $S$ and $T$, either of which, joined to $O$, fulfills the conditions.

Completing only the $S$-solution, draw $So$, and in $SoQ$ we have the angle $C$.

Draw the line $syE$ perpendicular to the edge $OQ$; revolve $Ss$ to $ss_1$ about $sy$ as an axis; then by joining $s_1$ with $y$ and carrying it by an arc, centre $y$, to $E$ on $sy$ produced, we shall have in $OE$ the second edge of face $c$.

Fig. 354.




$Os$ being the plan of the united $OR$ and $OR_1$, make $tr$, at 90° to it, for the h.t. of the plane containing angle $B$.

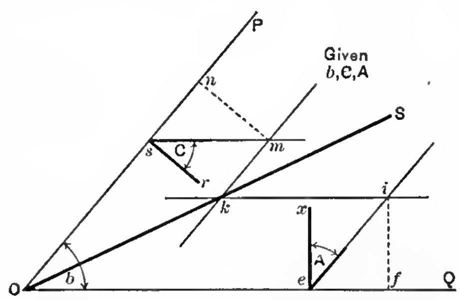
As before, drop perpendiculars $rp$ and $xz$ on the faces $c$ and $a$, and use them as radii with which to get $l$, the rabatted vertex of angle $B$. Then $lx$ and $lr$ complete the solution.

530. As earlier stated, the remaining cases may by means of the polar triangle be made to take the form of those already solved, although they are solved in the following articles without recourse to that expedient.

531. *Given, one side and the adjacent angles.*

(1) *Solved upon but one plane*, H, let $b$, the given side (Fig. 355) be taken in the plane of projection used. At any point of $OP$, as $s$, draw $sr$ at 90° to $OP$, and $sm$ making the given angle $C$ with $sr$. Similarly, at a random point $e$ of $OQ$, lay out the other given angle, $A$.

Fig. 355.

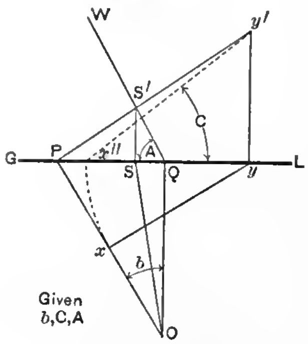



In $sm$ and $ei$ we see lines of declivity of the desired faces $a$ and $c$, after rotation into the plane of $b$.

Take some point $m$ on $sm$. When in its space-position the height of $m$ above the face $b$ is $sn$. Make $ef = sn$ and find $i$, a point of equal height with $m$ but on face $c$. Then $ik$ and $mk$ are the projections of *horizontal* lines in the faces $c$ and $a$, and being at the same level must intersect on the line of intersection of those faces; hence join $O$ with $k$ to obtain the plan of the space-edge. Solve the remainder as in earlier problems.

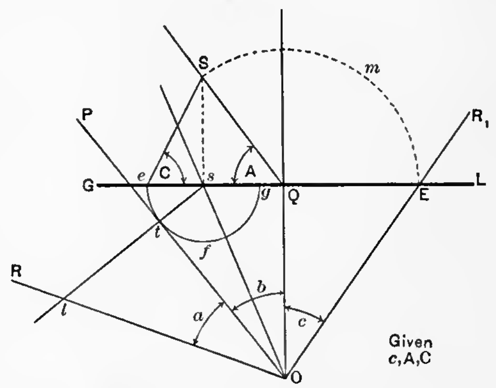
(2) *Solved with the use of both* H *and* V we would draw (Fig. 356) the face $b$ in H, with one edge $OQ$ perpendicular to G. L. Make angle $WQP = A$. Draw $xy$, a random line of declivity of the face $a$; carry $x$ to $x''$ (centre $y$), and draw from $x''$ a line making angle $C$ with G. L. This meets a vertical from $y$ at $y'$, one point of the trace of a plane making angle $C$ with H. $Py'$ is then the v. t. of face $a$. It crosses $WQ$ at $S'$, which—with $O$—gives the common edge of faces $a$ and $e$.

Fig. 356.




532. *Given, two angles and the side opposite one.* In Fig. 357 draw first the given side $e$, and at some point $Q$ draw $QE$, at 90° to $OQ$, to represent the plane of rotation of $E$. On this line lay off the given angle $A$, and find $S$ by arc of radius $QE$; then derive $s$ from $S$ as heretofore and join with $O$ for the plan of the space-edge of faces $a$ and $c$. Through $S$ draw a line $Se$ at an angle $C$ to $QG$. Using $Se$ as the generatrix of a vertical cone having $Ss$ for its axis, $e$ would trace the arc $etfg$; and $OP$, tangent to the latter, would be a trace of face $a$. Then $tl (= Se)$ gives $l$, through which draw $OR$ for the outer edge of face $a$.

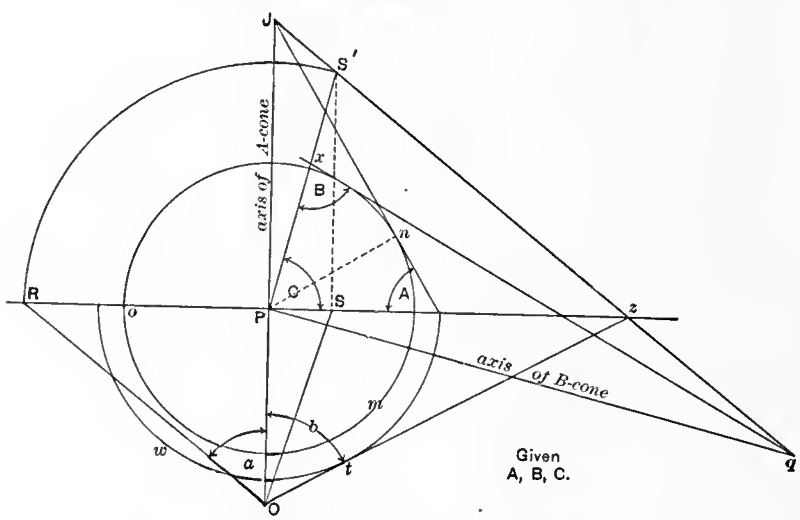
Fig. 357.




533. *Given, the three angles.* In Fig. 358 take $Rz$ for the ground line, and $OP$ at 90° to it for the edge common to $a$ and $b$. Draw $PS'$ at angle $C$ to $Pz$. Then $S'PO$, perpendicular to the vertical plane, is the face $a$ in its space-position.

We have next to adapt the problem of Art. 319, which required a plane whose inclinations to two mutually perpendicular planes were given. In the present case the planes $a$ and $b$ are not at 90° to each other; but the same principle applies, that by resting a plane against two cones having $A$ and $B$ for basal angles and whose axes intersect, each axis being perpendicular to one of the given planes, the conditions are met.

Fig. 358.




Let $P$ be the point of intersection of the axes of the auxiliary cones, and $JP$ the axis of a vertical cone whose base angle equals $A$. All planes tangent to this $A$-cone will be at the constant distance $Pn$ from $P$, but will contain the vertex $J$, and make the desired angle with the face $b$. Since $OPS'$ is perpendicular to V, an axis at 90° to it and containing the point $P$ will lie, like the other, in V, and is seen in the line $Pq$. Tangent to a circle (or *sphere*) of radius $Pn$ draw a line $xq$ at angle $B$ to $PS'$. It meets $Pq$ at $q$, the vertex of the $B$-cone. Then, as the desired plane must contain both vertices, $Jq$ is its v. t., and $ztO$, tangent to the base of the $A$-cone, is its h. t.

## *CHAPTER XII.*

ORTHOGRAPHIC.—STEREOGRAPHIC.—GNOMONIC.—NICOLISI'S GLOBULAR.—DE LA HIRE'S METHOD.—SIR HENRY JAMES' METHOD.—MERCATOR'S.—CONIC.—BONNE'S METHOD.—RECTANGULAR POLYCONIC.—EQUIDISTANT POLYCONIC.—ORDINARY POLYCONIC PROJECTION.

534. In attempting to represent the whole or any portion of the earth's surface or the celestial sphere upon a sheet of paper the draughtsman is confronted with the impossibility of avoiding a distortion of some kind, owing to the fact that he is dealing with a double-curved surface, which can not be "developed" or directly rolled out upon a plane. He is, therefore, compelled either to adopt some one of the many methods of applying the principles of perspective projection to the problem, or one of the equally large number of methods which, while not true projections in the ordinary sense, are now included under that head, owing to the extended mathematical significance of the term. In either case he will find that the system has been devised with a view to preserving between the original surface and its representation some relation which may be regarded as essential for the purpose for which the map or chart is to be used, but which can usually be attained only at the sacrifice of some other relation which it would be desirable to maintain. Thus, if he preserves upon his drawing the equality of the angles between the planes of the various circles that are usually represented, he cannot at the same time have all areas reduced in a constant ratio; and other desirable conditions are often found to be as mutually exclusive.

For an extended mathematical treatment of the various systems invented for the representation of the earth or the celestial sphere, as also for the tables essential to the construction of maps, the student is referred to Germain's *Traité des Projections* and Craig's *Treatise on Projections*, which, with De Morgan on *Gnomonic Projection*, have been the principal sources from which the writer has drawn.

Adopting Craig's classification, we group all the methods under the following heads:

(a) *Orthomorphic Projections*, in which similarity of *form* is secured between areas on the sphere and on the map.

(b) *Equivalent Projections*, in which different *areas* are reduced in the same proportion.

(c) *Zenithal Projections*, in which all points on the sphere that are equidistant from the assumed centre of projection are projected in a circle whose centre is the projection of the assumed perspective centre.

(d) *Projections by Development*, in which the spherical surface is first represented upon a tangent or secant developable surface, and the latter then rolled out upon a plane.

535. *The great circles customarily represented* are the *equator*, the *ecliptic* and the *meridians*.

*Meridians* contain the poles of the sphere, and their planes are perpendicular to that of the equator.

The *ecliptic* is the apparent path of the sun, and its plane makes an angle of $23°27'$ (usually called $23\frac{1}{2}°$) with the equator, cutting the latter at the *equinoctial points*.

The *Equinoctial Colure* is the meridian containing the equinoctial points.

The *Solstitial Colure* is a meridian whose plane is perpendicular to that of the equinoctial colure. It cuts the ecliptic at the *solstitial points*.

536. *The small circles usually projected* lie in planes parallel to that of the equator, and are, in particular, the *tropics* and the *polar circles.*

*The Tropics of Cancer and Capricorn* are the northern and southern limits, respectively, of the torrid zone, and being 23½° from the equator they touch the ecliptic at the solstitial points.

*The Polar Circles* are the *Arctic* and the *Antarctic*, each 23½° from a pole of the earth.

537. The *axis* of any circle is a straight line through its centre and perpendicular to its plane. It meets the surface of the sphere in points called the *poles* of the circle.

538. The *polar distance* of a *circle* is its distance from either of its poles, measured on the arc of a great circle. The plane on which the projection is made is frequently called the *primitive.* If it contains the centre of the sphere it cuts the surface in the *primitive circle.*

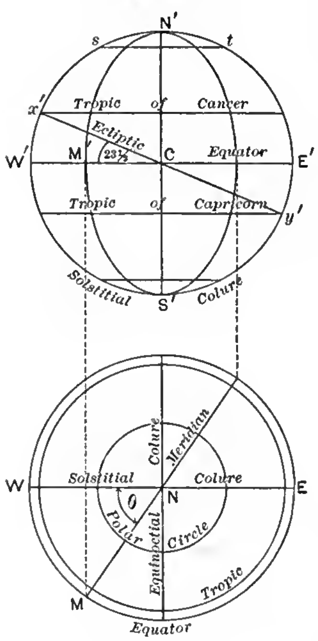
539. The *line of measures* of a circle is the intersection of the primitive plane by a plane containing the axes of the given and primitive circles, and is perpendicular to the intersection of those planes.

540. *The point of sight* (centre of projection) may be either *at infinity*—giving *parallel*, and, ordinarily, *orthographic projection*—or at a finite distance, giving a *perspective* or *central projection.*

### ORTHOGRAPHIC PROJECTION.

541. *Orthographic projection*—a special case of zenithal—is believed to have been first applied to the sphere by Hipparchus. It is not used for terrestrial maps, but solely for celestial charts.

Fig. 359.




In comparison with other systems orthographic projection has relatively greater distortions, both of form and area, for those portions of the sphere near the plane of projection. The inconvenience of constructing elliptical projections is another objection to it.

To show how the principal lines would appear by this method Fig. 359 is presented, the *elevation* having for the primitive the plane of the solstitial colure, that being the one usually selected for celestial charts constructed by this method. In this case all meridians appear as ellipses, while parallels are projected in straight lines.

The derivation of meridians from the plan is obvious, each appearing on the latter as a diameter.

*The equinoctial colure* is projected in the straight line $N'S'$, and the equinoctial points at $C$.

The *solstitial points* are seen at $x'$ and $y'$.

542. *Orthographic equatorial projection*, that is, with the equator as the primitive circle, is illustrated by the plan in Fig. 359, the *meridians* being, as already stated, *diameters*, while the *parallels* project in *concentric circles.*

543. The orthographic projection of the sphere upon the plane of a meridian or other circle making any given angle with the equator may be most readily obtained by the method of rotations (auxiliary planes) treated in Art. 404.

### STEREOGRAPHIC PROJECTION.

544. This projection, called by the above name only since 1613, was devised about 130 B. C. by Hipparchus, who called it a *planisphere.* It is an orthomorphic projection, and from the fact that it not only possesses the distinctive property of preserving similarity of form between infinitesimal

surfaces and their projections, but also that all circles on the sphere are projected as circles, it is the most convenient system so far devised, and at present the most employed for both geographical and astronomical purposes.

The eye may be located at any point *on the surface of the sphere*, but is usually taken either at some point on the equator, giving a *meridional* projection, or else at one of the poles, when an *equatorial* stereographic projection results.

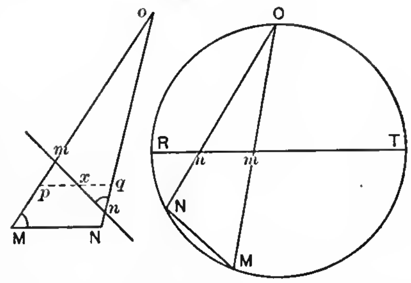
The projection is always made upon the plane of that great circle whose pole is the assumed point of sight.

Only the hemisphere opposite to the eye is projected.

545. The fact that every circle is projected, stereographically, into a circle, has been already established in Arts. 135–6, but for convenience the demonstration is repeated here.

In Fig. 360, regarding at first only the triangle on the left, let $o M N$ be a view of an oblique cone having a circular base $M N$.

Fig. 360.




Take a section plane $m n$ so as to make with $o N$ an angle $m n o$ equal to the angle $o M N$. Then will $m n$ be circular. Such a section is called *sub-contrary*.

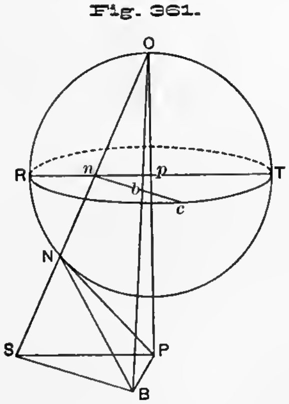
Take any section $p q$, parallel to $M N$, which will obviously be circular. From the similar triangles $p m x$ and $n x q$ we have $p x : n x :: m x : x q$, whence $p x \times x q = m x \times n x$. But $p x \times q x$ equals the square of the semichord common to the two sections at $x$; hence $m n$ is circular.

Turning now to the sphere $O R M T$, let $R T$ be the primitive circle (plane of projection), and $O$ (its pole) the centre of projection or position of the eye. Let $N M$ be any circular section of the sphere. Then $O N M$ is a visual cone, and $n m$, the projection of $N M$, will be a circle, being a sub-contrary section of the cone.

546. To establish the orthomorphic property of a stereographic projection we have to show that the angles between the projections of circles equal those between the original curves.

In Fig. 361 let $O$ be the position of the eye, and $P N$ a tangent at $N$ to the circle $O R N T$.

Fig. 361.




Then $n p$ is the projection of $N P$.

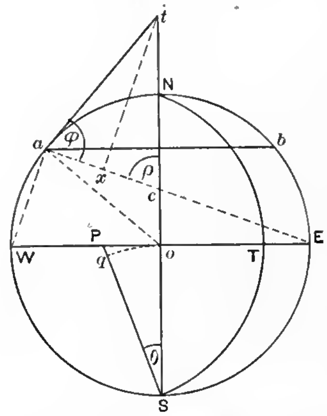
Let $B N$ be the tangent at $N$ to some other circle containing that point, and let $n c$ be the trace of the plane $O N B$ on the plane of projection $R c T$; then will the angle $p n b$ be equal to the angle $P N B$ between the tangents, i. e., the angle between the planes of the circles to which they are tangent. For we may take some point $S$, on $O N$ prolonged, and draw $S P$ parallel to $R T$, and $S B$ parallel to $n c$; then the plane of $S P B$ is parallel to the primitive, and the angle $P S B$ obviously equal to $p n c$. The angle $O n T$, being between chords, is measured by $\frac{1}{2}(O T + R N)$; but this equals $\frac{1}{2}(O R + R N)$ which measures the angle between tangent $P N$ and chord $N O$. Hence $O n T = O S P = P N S$; whence $P N = P S$. Similarly, we may prove $B N = B S$. The triangles $B N P$ and $B S P$ have the sides of the one equal to those of the other, each to each, and being, therefore, equal in all their parts, we find angle $B N P = B S P = p n b$.

Since a curve and its tangent are always projected as a curve and a tangent, and since the angle between two circles is that between their tangents at a common point, the proposition is established.

547. *Stereographic meridian projection.* As usually constructed, the plane of projection is that of the meridian of Greenwich, the actual longitude of the eye being then either 90° or 270°. The meridian through the eye is, however, called the first meridian, and graduation made therefrom each way, from 0° to 90°.

*To project a parallel of latitude*, as $ab$ (Fig. 362), draw a *tangent* at either extremity, as $a$, to meet the radius $oN$ prolonged. This tangent, $ta$, is the radius of the arc $acb$ in which the parallel is projected.

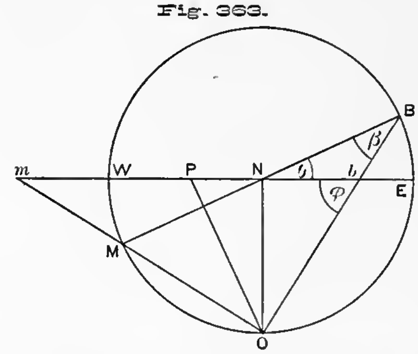
Fig. 362.




For imagine the sphere rotated 90° to the right, so that the eye moves from $o$ to $E$, and $NS$ becomes the projection of the primitive. The projector $Ea$ cuts the primitive at $c$, which remains constant during counter-rotation. Then the centre $t$, of arc $acb$, must be at the intersection $t$ of $oN$ by a perpendicular to chord $ac$ at its middle point. In the triangles $atx$ and $txc$ we have the angles $\rho$ and $\phi$ equal, being opposite equal sides. But $\rho$ is the angle between two chords, and is measured by $\frac{1}{2}(SE + aN) = \frac{1}{2}(NE + aN) = \frac{1}{2}\,aNE$, which, as the measure of $\phi$, shows that if $aE$ is a *chord*, $at$ must be a *tangent*.

*To project a meridian of longitude.* Let the meridian to be projected make an angle $\theta$ with the plane of the primitive meridian, $WNES$, upon which it is to be projected. Draw $SP$ at $\theta°$ to $SN$; then $P$ is the centre and $PS$ the radius of the arc $NTS$ in which the meridian projects. This is established as follows: Let Fig. 363 represent a top view of a sphere; $O$ the point of sight on the equator; $MB$ the plan of a meridian making $\theta°$ with the primitive, $mWE$; then $mb$ is the plan of the circle in which $MB$ is stereographically projected; and $P$, bisecting $mb$, is the centre of the projection. Now as $MOB$ is a right angle we have $PO = Pb$; hence angle $POB$ equals $PbO$, or $\phi$. But $\phi = \theta + \beta$; and as we have $NOB = \beta$, it follows that $PON$ equals $\theta$.

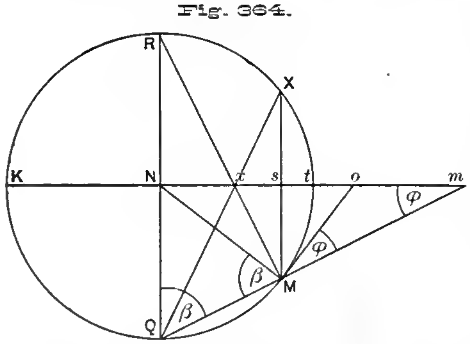
Fig. 363.




Draw the arc $oq$ in Fig. 362. Then $P$, the centre of the meridian $NTS$, is distant from the centre of the sphere an amount $oP$ which is the *tangent* of the inclination of the meridian to the primitive; while $SP$, the *radius* of the projection, is the *secant* of such inclination.

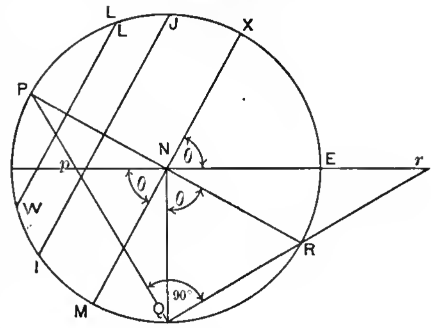
548. *To project a small circle stereographically when its plane makes an angle of 90° with the primitive.* In Fig. 364 let $XM$ be a small circle whose plane is perpendicular to the primitive $KNm$; then the projectors $QX$ and $QM$ give $xm$ for the stereographic projection as seen in plan. Since $RX$ equals $QM$ the lines $RM$ and $QX$ will meet at $x$. Bisect $mx$ at $o$. Draw $Mo$ and $MN$. The angle $RMm = RMQ = 90°$; hence $oM = ox = om$. The angles $\phi$ are equal; also angles $\beta$. We then have $NQM + NmQ = NMQ + oMm = 90°$; hence $NMo = 90°$, and $Mo$ is a tangent to arc $Mt$. We see, then, that $oM$, the *radius* of the projection, equals the *tangent* of the polar distance $Mt$; while the centre $o$ of a small circle's projection is distant from the centre $N$ of the primitive an amount $No$ equal to the *secant* of said polar distance.

Fig. 364.



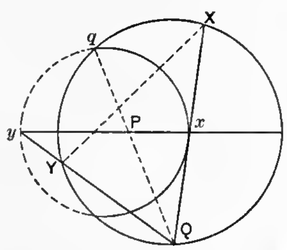
549. If $\theta$ is the angle between the primitive and any circle, great or small, the poles of such circle will project upon the line of measures at distances which are, respectively, $\tan\frac{\theta}{2}$ and $\cot\frac{\theta}{2}$, the former for the pole farthest from the centre of projection. For in Fig. 365, which is a plan of the sphere, $Q$ is the centre of projection; $pNE$ the primitive; $P$ and $R$ the poles of circles $WL$, $IJ$, $MX$, all inclined $\theta°$ to the primitive; $p$ and $r$, the projections of the poles, and $Np$ and $Nr$ to have values as just stated. The angle $NQr$ is 90°. Arc $QR$ equals $EX$. Hence $QPR = \frac{1}{2}\theta$, and $pN = \tan\frac{1}{2}\theta$. $Nr$ obviously then equals $\cot\frac{1}{2}\theta$.

Fig. 365.




550. *Having given the pole of a circle, to project the circle.* Let $P$ (Fig. 366) be the pole of a *small* circle whose polar distance is 60°. Prolong $QP$ to $q$ and lay off arcs $qX$ and $qY$ each equal to 60°. Project $X$ at $x$ and $Y$ at $y$ and draw a circle on $xy$ as a diameter. This will be the projection sought.

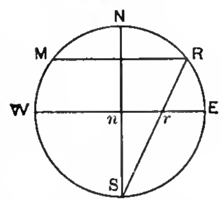
Fig. 366.




For a *great* circle the arcs $qX$ and $qY$ would be made 90°, but otherwise the solution would be unchanged.

551. *Stereographic equatorial projection.* In this projection the meridians project as straight lines, and the parallels as circles, concentric with the primitive.

Fig. 367.




The radius of any projection will be the tangent of one-half the polar distance. Thus, in Fig. 367, let the circle $WNES$ represent the equator and let $MR$ be a parallel of latitude. We shall then have $nr$ equal to the tangent of one-half the arc $NR$.

552. Although distortion of *form* is obviated by using stereographic projection, that of *areas* is quite considerable near the centre of the map as compared with the outside.

GNOMONIC.—NICOLISI'S GLOBULAR.—DE LA HIRE'S.—SIR HENRY JAMES'.

553. *Gnomonic projection* is believed to have been the first method employed in projecting the sphere, it dating back to Thales, one of the seven wise men of Greece. Although used chiefly for celestial charts, it derived its present name from its serviceability in the graduation of gnomons. It has been employed to some extent for representing the polar regions. This projection is made upon a tangent plane to the sphere, the eye being taken at the centre of the latter.

Every great circle will project in a straight line, while small circles parallel to the primitive will project in concentric circles whose radii are the tangents of their polar distances.

As the great circle parallel to the primitive will project at infinity, this method will evidently not answer for an entire hemisphere.

554. *Nicolisi's Globular Projection.** This method of representation, for it is not a true projection, is largely employed in making terrestrial maps.

As meridians and parallels appear as circular arcs, it has in that respect the same advantage as stereographic projection over others less conveniently constructed. It lacks, however, the orthomorphic property of the stereographic. It was invented in 1660 by J. B. Nicolisi, of Paterno, Sicily.

To represent the meridians and parallels by this method draw a circle for the *primitive meridian;*

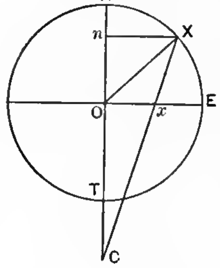
*Following Germain, the term *globular* is here applied to the method which first received the name. Having been introduced into England in 1794 by Arrowsmith it has been erroneously accredited to him.

a horizontal diameter for the projection of the equator, and a vertical diameter for that of the central meridian. Then, for ten-degree intervals, divide the quadrants into *nine* equal parts, numbering each way *from the equator.* Also divide the horizontal and vertical radii into *nine* equal parts, numbering each way *from the centre.* The *parallels* are then drawn as circular arcs through like-numbered divisions each side of the equator, while the *meridians* are circular arcs containing the poles and the divisions on the equator.

555. *De La Hire's Perspective Projection.* This projection, often erroneously termed globular, was devised by De La Hire in 1701. In it the eye is taken on the prolonged axis of the primitive, and at a distance from the surface of the sphere equal to the sine of 45°. Its classification is obviously under zenithal projection.

With a radius of unity the sine of $45° = \sqrt{\tfrac{1}{2}}$. Hence, in Fig. 368, if $CT = nX = \sqrt{\tfrac{1}{2}}$, then $Ox$ and $xE$, which are the projections of the 45°-arcs $NX$ and $XE$, will be equal.

Fig. 368.




Other equal arcs will have projections very nearly equal. This is its only practical advantage, as it is neither orthomorphic nor equivalent, and involves elliptical projections for all circles not parallel to the primitive.

556. *Sir Henry James' Perspective Projection* is an interesting case of zenithal, devised for the purpose of reducing the misrepresentation to a minimum. Like De La Hire's, the eye is taken exterior to the sphere, but in this case at a distance equal to one-half the radius.

For a hemisphere this is regarded as the best possible system of projection. By taking a plane of projection parallel to the ecliptic and touching one of the tropics, or, in other words, by adding a $23\tfrac{1}{2}$°-zone to the hemisphere, Colonel James obtained America, Europe, Asia and Africa in one projection, claiming it to include "two-thirds of the sphere." This has been shown by Captain A. R. Clarke to be an underestimate, the exact figures being *seven-tenths;* while the same writer shows that for minimum distortion with the new primitive the eye should be at a distance of $\tfrac{11}{30}r$ outside the surface, instead of $\tfrac{15}{30}$.

PROJECTION BY DEVELOPMENT.—CYLINDRIC.—CONIC.—POLYCONIC.

557. With the eye at the centre of the sphere we may project the various circles of the latter upon either a *cylinder* that is tangent or secant to the sphere, or upon a tangent or secant *cone.* By then developing the auxiliary surface we will have in the one case a *cylindric* and in the other a *conic* projection.

558. *In square cylindric projection* the auxiliary cylinder is tangent along the equator. The *meridians* then appear as *straight lines* perpendicular to the rectified equator, while the *parallels*—which *projected* as *circles*—*develop* into *straight lines* at 90° to the meridians, the distance of each from the equator being the tangent of its latitude.

This projection is only occasionally used, the exaggerations involved being too great to make it serviceable except for a short distance each side of the equator.

559. *Mercator's projection* (also called a *reduced chart*) differs from the last described only as to the spacing of parallels. This spacing is, however, so effected that on the resulting map the angles are preserved between any two curvilinear elements of the sphere; in other words, Mercator's is an orthomorphic projection.

Since *meridians* actually converge on a sphere at such rate that the length of a degree of longitude at any latitude equals that of a degree on the equator multiplied by the cosine of the latitude, it is obvious that when they are represented as *non-convergent* the distance apart of originally

equidistant parallels of latitude should *increase* at the same rate; or, otherwise stated, as on Mercator's chart degrees of longitude are all made equal, regardless of the latitude, the constant length representative of such degree bears a varying ratio to the actual arc on the sphere, being greater with the increase in latitude; but the greater the latitude the less its cosine or the greater its secant; hence lengths representative of degrees of latitude will increase with the secant of the latitude.

The increments of the secant for each minute of latitude can be ascertained from tables.

Navigators' charts are usually made by Mercator's projection, since upon them (as upon the square cylindric) *rhumb lines* or *loxodromics*—the curves on a sphere that cross all the meridians at the same angle—are represented as straight lines.

A loxodromic not being also a geodesic, the mariner takes for his *practical* shortest course between two points the portions of those different loxodromics which most nearly coincide with the great-circle arc through the points.

560. *In conic projection*, if the auxiliary cone be tangent along a parallel of latitude, the *meridians* will project as *elements* of the cone; the *parallels* into *circles*. On the *development* the parallels become *concentric arcs* on the sector into which the cone develops, the radius of each being the slant-height distance from the parallel to the vertex. The *meridians* obviously develop into *radii* of the sector.

561. If tangent to the sphere near the equator the vertex of a cone is inconveniently remote. Even when tangent along a parallel of latitude more medially situated this method gives undue distortion, except for a narrow zone on which the parallel of contact is central. Many methods have been devised for the purpose of obviating these difficulties, a few of which are next briefly mentioned.

562. *Mercator* suggested the substitution of a *secant* for a *tangent cone*, choosing its position with reference to the balancing of certain errors. By this method a large map of Europe was made in 1554. *Euler* carried out the same idea with greater exactness, fulfilling his self-imposed conditions that the errors at the northern and southern limits should not only equal each other, but also the maximum error near the mean parallel.

563. *Bonne*, in 1752, applied the following method (its invention is variously accredited) which was later adopted (1803) by the French War Department, and has been extensively used in European topographical work: Assuming a central meridian and a central parallel, a cone is made tangent to the sphere on the parallel. The central meridian is then rectified on the element tangent to it, and using the cone's vertex as a centre circular arcs are drawn through (theoretically) consecutive points of the developed meridian. The zones between the consecutive parallels on the sphere then develop in their true areas upon a plane. Each meridian is drawn upon the map so as to cut each developed parallel at the same point as on the sphere. The parallel of tangency cuts each meridian at a right angle.

Bonne's method evidently comes under the head of *equivalent* projections, as it preserves the *area* though not the *form* of all elementary quadrilaterals.

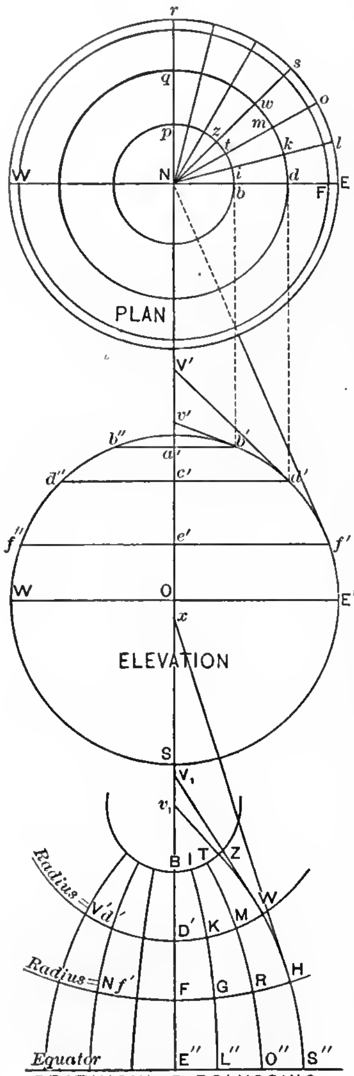
When extended to include the whole earth in one view the map has a peculiar shape, somewhat like a crescent with full, rounded ends, and quite broad at the centre.

By taking the equator for the "central parallel" a projection results, due to *Sanson*, called *sinusoidal* by *d'Avezac*, and often credited to *Flamstead*. When applied to the entire sphere it resembles two equal and opposite parabolas with their extremities joined.

564. *Polyconic Projection*. A method largely used in England and employed by the United States Government on its Coast and Geodetic Survey, is based upon the use of a separate tangent cone for each parallel to be developed.

565. *In Rectangular Polyconic Projection* the rectangularity of the quadrilaterals between meridians and parallels is preserved. It is thus constructed: In the elevation, Fig. 369, let $b''b'$, $d''d'$, $f''f'$ be parallels of latitude; their plans will be the concentric circles shown in the upper figure. $Nf'$, $V'd'$, $v'b'$, are the elements of cones, tangent to the sphere on the indicated parallels.

Fig. 369.



RECTANGULAR POLYCONIC

Taking the meridian $E d b N$ as the central meridian, rectify it at $E'' D B$ in the lower figure, getting the lengths from $E' d' b'$. Make $E'' F = E' f'$, etc.

Through $F$ an arc of radius $Nf'$ is the development of the parallel $f''f'$. The arc through $B$ has radius $v' b'$.

On the plan draw meridians $Nl$, $No$, etc. Then lay off on each developed parallel the distances included on it between the meridians just drawn. Thus, $D K$ and $K M$ equal the rectified arcs $d k$ and $k m$. $B T Z$ equals the true length of $b t z$.

When the parallel of tangency is so near the equator as to make the vertex of the auxiliary cone inconveniently remote, tables are employed giving the rectilinear coördinates of points on the developed parallels.

566. *Equidistant Polyconic Projection* is a modification of the method just described, resulting in a representation in which two parallels will include equal arcs on all meridians.

This method is used in Government work, for small areas. To draw it a central meridian $E'' F D B$ (Fig. 369) and a central parallel $D K M W$ are drawn as in the rectangular polyconic system, and the meridians also found in the same manner, or by the use of tables. From the points $D$, $K$, $M$, $W$, where the central parallel intersects the meridians, the equal lengths $F D$, $D B$, are laid off on the meridians, giving points through which the other parallels may be drawn.

567. *Ordinary Polyconic Projection.* This method sacrifices the rectangular intersection of meridians with parallels (except on the central meridian) in order to preserve the lengths of the degrees on the parallels.

Drawing the usual central meridian in its true length, the parallels are developed as for the rectangular polyconic; but on each parallel the degrees of longitude are laid off in their actual lengths, and points thus obtained through which to draw the meridians.

This method is in general use by the U. S. Government for the maps of its Coast Survey.

568. The foregoing is as extended an excursion into this attractive field as the limits of this treatise will permit, but it should be understood to be but a glance, and that a large number of interesting methods must go unnoticed, the student being referred to the authorities earlier mentioned, in case he wishes to pursue the subject further.

# CHAPTER XIII.

## SHADES AND SHADOWS OF MISCELLANEOUS SURFACES.

569. The shadows cast by an object which is illumined by either the sun or some other source of light are, in the mathematical sense, *projections*, and the rays of light become the *projectors*.

570. The *shade* of an object is that part of its own surface which receives no direct rays from the source of light, while the *shadow* is the darkened portion of some other surface from which the original object excludes the light.

The rays through all points of a given line will determine either a *plane of rays* or a *cylinder of rays*, according as the line is straight or curved.

571. The *line of shade* on an object is the boundary between the illumined and the unillumined portions, and its shadow forms the boundary of the shadow cast by the object.

If the object is curved, the line of shade is the line of contact of a tangent *cylinder* of rays, each element of which would be tangent to the object at a point at which the cylinder and the object would have a common tangent plane of rays.

Fig. 370.

For convex plane-sided surfaces the line of shade is the warped polygon formed by the edges contained by non-secant planes of rays.

572. It is the province of Descriptive Geometry, in its application to this topic, merely to determine the rigid outlines of shadows and shades. The delicate effects of cross and reflected lights, which always exist in nature in greater or less degree, can only be theorized about in a general way and can be most successfully imitated in draughting by working from a model or by the aid of photographs.

Fig. 371.

Figs. 370 and 371, which are half-tone reproductions of photographs of plaster models, while illustrative mainly of shades as distinguished from shadows, also show the absence on double curved surfaces of those rigid lines of demarcation to which theoretical constructions lead. Yet the ability to correctly locate the geometrical lines of shade and boundaries of shadows is as essential an element of the draughtsman's education as a knowledge of the laws of perspective; since, lacking either, he could neither make an intelligent visit to an art or architectural exhibition nor so work up an original design that it could bear critical examination.

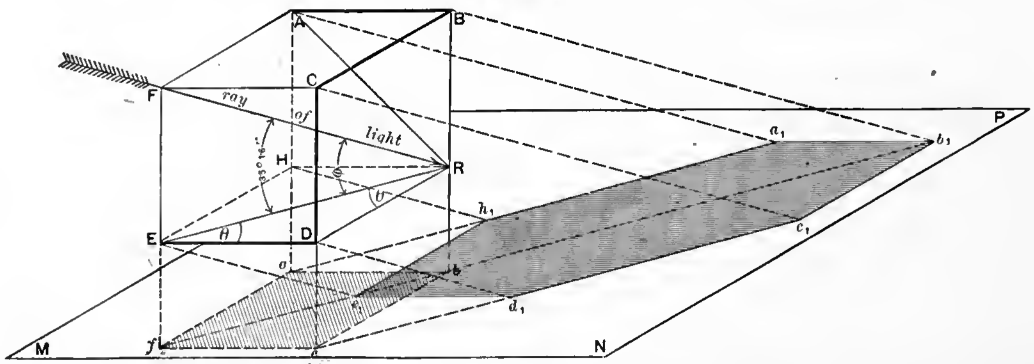
573. A conventional rule, much employed for throwing *machine drawings* into sharp relief, is to

make the right-hand and lower boundaries of a flat surface shade (i.e., *heavy*) lines, provided that they separate visible from invisible surfaces. When, however, they are located with reference to some source of light, as in architectural and other drawings, the shade lines are those which could cast shadows, and their determination usually requires more thought, as some of the succeeding problems will show.

574. *Direction of Light.* A direction quite often (though not necessarily) assumed for the light is that of the body-diagonal of a cube whose faces are parallel to the planes of projection; that diagonal, in particular, which descends from left to right in approaching the vertical plane. It is illustrated by the arrow in Fig. 372, the source of light being assumed to be the sun, whose rays may for all practical purposes be regarded as parallel.

575. The ray $FR$ (Fig. 372), being the body-diagonal of the cube, will project in $ER$ on the

Fig. 372.




base, or in $AR$ on the back. $ER$ and $AR$, being diagonals of squares, make 45° with the edges of the cube, which has led to the expression "light at forty-five degrees," for the conventional direction. But the ray itself makes an angle ($\phi$) of 35° 16′ with either $ER$ or $AR$, as may be established thus: Taking the edge of the cube as unity we have $ER = \sqrt{2}$; also $\tan\ \phi\ (ERF) = 1 \div \sqrt{2}$, which, in a table of natural tangents, corresponds to the value indicated.

Shadows thus cast are seen to be true oblique or clinographic projections (Art. 14), although orthographic projections are usually employed as auxiliaries in their determination.

576. In pictorial illustration of a few general principles Fig. 372 is drawn in oblique projection. In so elementary a figure as the cube the edges to be drawn as shade lines are evident from the outset, since, for the given direction of rays, the back, right and lower faces are obviously in the shade. We proceed then directly to find the shadow of the warped polygon $ABCDEHA$, establishing at the same time some of the principles that are of most frequent use.

577. *The shadow of a point upon a surface,* being the intersection of the surface by a ray through the point, is found where the ray meets its projection on the surface.

In Fig. 372 the point $C$ is projected on the plane $MP$ at $c$. The ray $Cc_1$ meets its projection $cc_1$ at $c_1$, which is, therefore, the shadow sought.

578. *Any line, straight or curved, is equal and parallel to its shadow, when the plane receiving the shadow is parallel to the line casting it.* Having $c_1$ we therefore draw $c_1b_1$ and $b_1a_1$ parallel to $CB$ and $AB$ respectively.

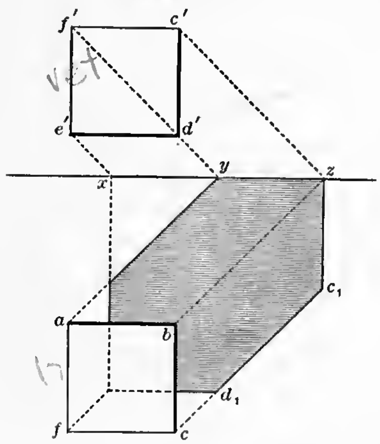
579. *A line that is perpendicular to a plane* will cast a shadow upon it whose direction is that of the projection of rays upon the plane.

The trace, $cc_1$, of the vertical plane of rays through $CD$, contains the shadow $d_1c_1$, and the projections of both rays $Cc_1$ and $Dd_1$.

580. *Parallel lines cast parallel shadows on a plane,* since they are the intersection of parallel planes of rays by a third plane.

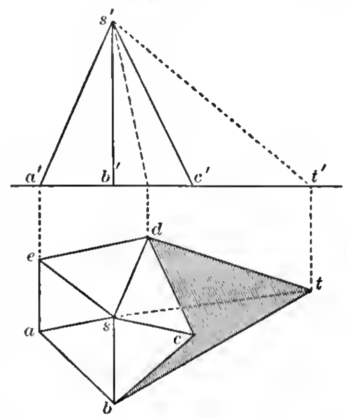
581. In orthographic projection the construction of Fig. 372 is shown in Fig. 373. The rays are shown on V in $c'x$, $f'y$ and $c'z$, and each projects to the proper plan to give the trace of a ray on H.

Fig. 373.




582. *The shade and shadow of a vertical pyramid. As the point where a line meets a surface is the beginning of the shadow of the line on the surface,* we have merely, in Fig. 374, to join $t$, the trace of the ray through the vertex, with $b$ and $d$, where the edges of shade ($s\,b$ and $s\,d$) meet H, to include the shadow sought.

Fig. 374.




Planes of rays through the other edges would in each case be secant to the pyramid, and therefore useless.

583. *Shadows on vertical, horizontal and oblique planes* are illustrated by Fig. 375, in which the pier flanking four steps receives on its inclined face the shadow of a vertical post, and in turn casts a shadow on the steps.

*The shadow of the post.* To find the point $y''$, $y_1$, where the ray through the vertex $y$ meets the face $a\,b\,c\,d$, regard $y\,B$ as the trace of a vertical plane through the ray; note $A$ and $B$, where it cuts bounding lines of the inclined surface; project these at $A'$ and $B'$, upon the elevations of the same edges, and draw $A'B'$ for the v. p. of the intersection of these planes. This receives the ray from $y'$ at $y''$, which projects down to $y_1$. A similar construction gives $u''$, which joins with $y''$ for the shadow of the edge $y'u''$.

For the shadow of edge $r\,y$, $r'y'$, we may regard the face $a'b'c'd'$ as extended upward and to the left, sufficiently for a re-application of the method described, $D'$ giving a *direction* for the line $y''D'$, which is a real shadow only to the edge $a'd'$.

The vertical edges of the post, whose plans are $u$ and $r$, cast shadows on H in the direction of the projection of rays.

*The shadow of the pier on the steps.* The vertical edge at $b$ casts a shadow beginning at the foot of the line, and running — in the direction of plans of rays — to $b_1$, the h. t. of the ray $b'N$. At $b_1$ the shadow of the line $b'c'$ begins, and its *direction* — upon horizontal planes — is found by treating $c\,c'$ as if it actually cast a shadow on the ground, $t$ then being the h. t. of the ray from $c'$, and $t\,b_1$ the direction sought. At $k$, however, the shadow begins to fall on the front of the lowest step, and we project up to $k'$ for its v. p. To get the direction of shadows on the fronts of the steps, assume on $b'c'$ some point $i$, $i'$; imagine the front of step No. 1 extended to catch the ray through $i'$ at $l'$, then toward $l'$ the shadow $M$ runs from $k'$. This direction once established, we have only to get one point of each shadow $P$ and $Q$ in order to draw them in; while $R$, $S$ and $T$ may be drawn parallel to $b_1t$, when one point of each is known.

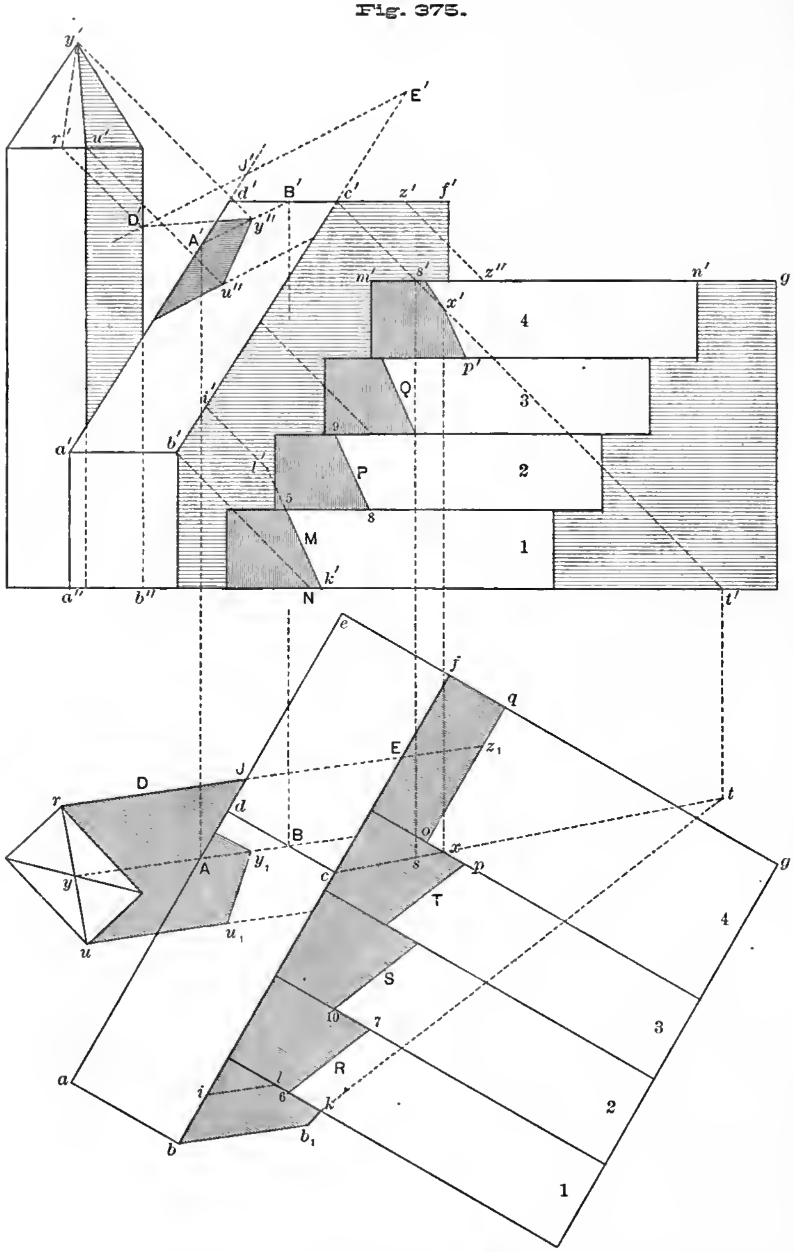
The shadow $M$ runs off the front of step 1 at 5, which projects down to 6 for the beginning of shadow $R$. The latter is parallel to $b_1t$, and at 7 meets the lower edge of the front of the second step. It projects to 8, through which the shadow $P$ is drawn parallel to $k'l'$.

To determine where this process will terminate we may definitely locate the shadow of $c\,c'$, either as a preliminary or at any stage of the work, thus: The ray $c't'$ meets the *level* of the top step at $s'$; this projects upon $c\,t$ at $s$, which is outside of the actual limits of step 4 and therefore

unreal. The plan $c\,t$ of the same ray meets the front of 4 at $x$, which projects at $x'$, and this, being between the limits of the front of the step, is therefore a real shadow.

At $x'$ the shadow has an angle, corresponding to that between $b'c'$ and $c'f'$. Its direction above $x'$ is most easily determined thus: Assume some point, as $z'$, whose shadow is likely to fall on the top step; find its shadow $z_1$, and through it draw the line $o\,q$ parallel to the line casting the shadow; then project $o$ from the front edge of the upper step up to $m'n'$, the v. p. of the same edge, and there join with $x'$.

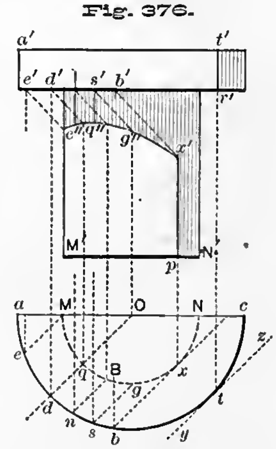
Fig. 375.



584. *The shadow of a cylindrical abacus upon a similarly shaped column.* Let $MBN$ (Fig. 376) be the plan of the cylindrical column, and $a\,b\,c$ that of the abacus. The vertical plane of rays $y\,z$ gives, by its tangency at $t$, the element of shade $t'r'$ on the abacus. A similar tangent plane of rays $b\,x$ gives the element of shade $p\,x'$ on the column. It contains $b\,b'$, that point of the abacus which casts the last shadow, $x'$, on the column. Any other vertical plane of rays, as $O\,d$, will cut a point $d\,d'$ from the abacus, and an element from the column, the latter catching the ray from the former at $q$.

The point last found is necessarily the highest in the shadow, as it lies in that plane of rays which contains the axis; in other words, the meridian plane of rays.

Fig. 376.



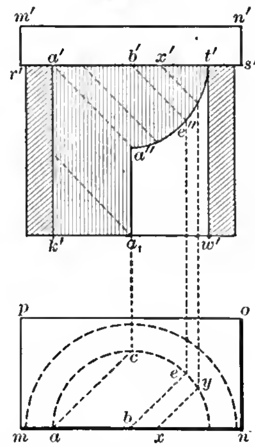
585. *The shadow of a rectangular block resting on a vertical semi-cylinder.* Let $m\,n\,o\,p$ (Fig. 377) be the plan of a block whose front coincides with the section-lined surfaces of the cylinder.

The edge $a't'$ will cast an elliptical shadow $a''e''t'$, which is the intersection of the inner cylindrical surface by the plane of rays through $a't'$. To find it regard $a\,c$, $b\,e$, $x\,y$ as traces of vertical planes of rays, as in the last problem; draw rays

$a'a''$, $b'e''$, etc., through the points casting shadows, and note where each ray meets the element lying in the same plane with it.

The shadow $a''a_1$ is cast by an equal length of $a'k'$, found by drawing a ray back from $a_1$. The remainder of $a'k'$ would cast the shadow $ac$ upon any horizontal plane on which the object might be regarded as resting.

Fig. 377.

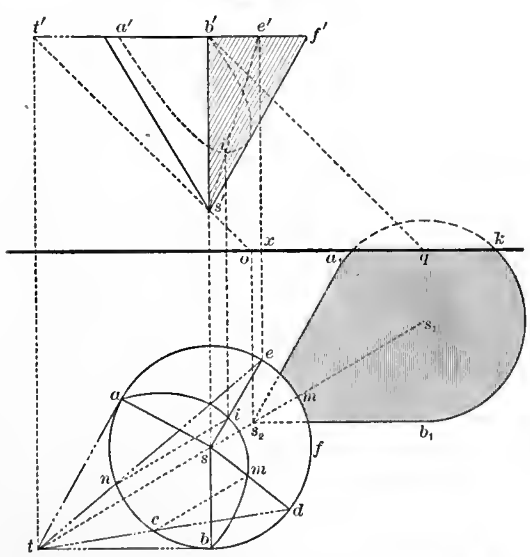


586. *The shade and shadow of a vertical, inverted, hollow cone.* In Fig. 378 let $a's'f'$, $abf$, represent the cone. The lines casting shadows will be the elements of shade and a portion of the base.

*The elements of shade* will be the lines of contact of tangent planes of rays. Each of these planes will contain the ray $st$, $s't'$, through the vertex, and will cut the plane of the base in tangents to the latter; hence from $tt'$ —the trace of the ray—draw $ta$ and $tb$ tangent to the base; then $as$ and $bs$ are the plans of the elements of shade. Of these the latter only is visible in elevation; and the surface $b's'f'$ on its right is the visible portion of the shade.

*The shadow on* H. The ray $t's'$ has its horizontal trace at $s_2$, one point of the shadow on H. As the arc $aedb$ must cast on H a shadow that is equal and parallel to itself (Art. 578) find $s_1$, the centre of the latter, by the ray from $b'(s)$; then the arc $a_1kb_1$, limited by tangents from $s_2$, completes the shadow on H.

Fig. 378.



*The shadow on the interior.* Any secant plane of rays through the vertex, as $tsd$, will cut a point $c$ from the base on the side toward the light, and an element $sd$ on the opposite side of the cone. The ray through $c$ will then intersect the element at a point $m$ which is on the limiting line $aib$ of the interior shadow. Other points are analogously found, as $i$ from $n$.

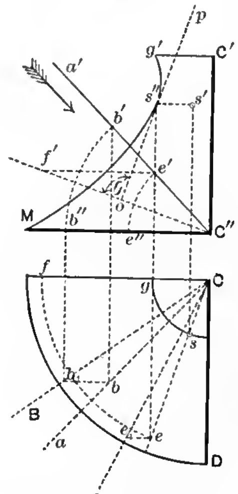
The shadow obviously terminates at the tangent points, $a$ and $b$.

587. *Brilliant points.* All are familiar with the marked contrast as to brilliance between the various portions of a highly polished surface. The point which seems brightest to the observer is, however, not actually the one receiving the light most directly, but is that from which the incident (direct) ray is reflected directly to the eye, in conformity with the well-known optical law that an incident ray and the same ray as reflected make equal angles with the normal to the reflecting surface. Since in orthographic projection the reflected ray is perpendicular to the paper, we proceed as follows to find a brilliant point: Obtain the bisector of the angle between a ray of light and a perpendicular to the paper; then pass a plane perpendicular to such bisector and tangent to the surface. Its point of contact will be the brilliant point sought.

588. *To find the brilliant point on a surface of revolution.* In illustration of the principles stated in the last article Fig. 379 is presented, in which one-quarter of a surface of revolution appears, $Ms''g'$ being its meridian curve. The direct ray through $C''C$ is $a'C''$, $aC$. The reflected ray from the same point is $CD$, $C''$. To find the bisector of the angle whose plan is $aCD$, carry the ray $a'C''$ into H, when it appears at $b_1C$, $b''C''$. $Ce_1$ bisects angle $BCD$, and in space becomes $e'C''$,

*e C*. The vertical meridian plane through *C e* may be rotated till parallel to V, when the bisector just mentioned will appear at $f'C''$, and the meridian curve will project in $M s'' g'$. Tangent to the latter and at 90° to $f'C''$ draw *o p*, which represents an edge view of a tangent plane. Its point of contact, $s''$, counter-revolves to $s'(s)$, the brilliant point desired.

Fig. 379.



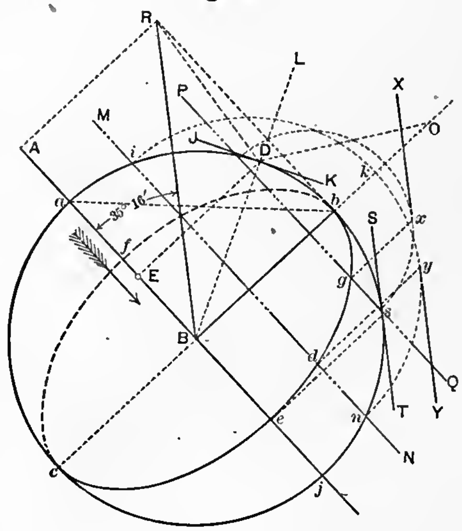
589. *The brilliant point and the curve of shade on a sphere.* In Fig. 380 the sphere is represented by one view only, its elevation, *a c j b;* the vertical plane of projection being understood to contain the centre, *B*.

*The brilliant point.* *A B*, parallel to the arrow, is the *projection* of the ray through the centre, the ray itself making an angle of 35° 16′ with the plane of projection (Art. 575).

The reflected ray from *B* is projected in that point. If the plane of the incident and reflected ray be rotated into V about *A j* as an axis, the former will appear at *R B* and the latter at *b B*. *L B* is the bisector of the angle *R B b*, and *D* therefore the rabatted brilliant point, *E* being its true position.

Were *J K* a tangent mirror and *R* a luminous point, the incident ray *R D* would be reflected to a point *O* on *B b*, at a distance from *B* equal to *R B*.

Fig. 380.



*The curve of shade.* This will be the circle of contact of a tangent cylinder of rays. It may be found by points, thus: Take a series of auxiliary planes of rays perpendicular to the paper, as *M N*, *P Q*, etc. Each will cut the sphere in a circle which may be seen as such by a 90°-rotation of the plane; a revolved ray can then be drawn tangent to the circle, and its true position found by counter revolution. In the auxiliary plane *M N*, for example, we have *i k n* for half the circle cut from the sphere, and a tangent thereto at *y*, by a ray parallel to *R B*, gives a point which, after counter-revolution, appears at *d* on the curve of shade *c e b f*. The point *e* is similarly derived from *s*, and *g* from *x*. Parallels to *A B* and tangent to the spherical contour at *a* and *b* give the extremities of the major axis of the ellipse in which the curve of shade projects.

590. *The curve of shade upon a torus.* In illustration the architectural torus (Fig. 381) is employed, although the same methods are applicable to the annular torus.

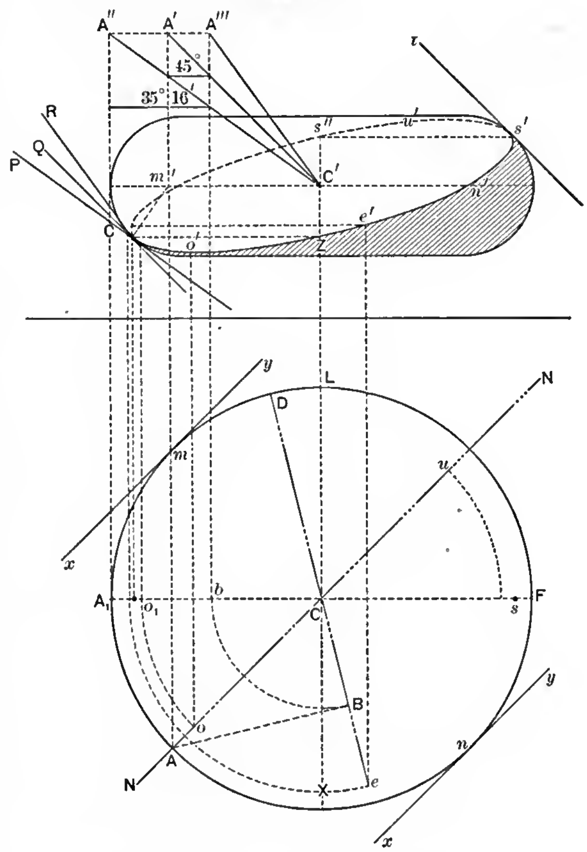
*The points on the "equator" of the surface*, viz., $m'$ and $n'$, are the points of contact of two vertical planes of rays, each indicated by *x y* on the plan.

*Points on the apparent contour of the elevation* are determined by the tangency of planes of rays perpendicular to V, as $t s'$ and that through *Q*, each at 45° to the horizontal.

*The highest and lowest points*, $o o'$ and $u u'$, *which must lie in the meridian plane of rays*, *N N*, are found by revolving the meridian section in that plane about the vertical axis through *C*, until parallel to V, when it will be projected in the given elevation. The ray *A C*, $A'C'$, being revolved at the same time, becomes $A_1 C$, $A''C'$, the latter then making 35° 16′ with H. Parallel to $A''C'$ draw tangents to the elevation (only one, *P*, drawn) and counter-revolve the contact points into the first position of the plane *N N*. They appear at *o* and *u*, from which the elevations are derived.

*Points in any assumed meridian plane.* Let $D\,C\,e$ be any meridian plane. Project the ray $A\,C$, $A'C'$ upon it at $C\,B$; rotate the plane to $A_1F$, when $A$ goes from $B$ to $b$ and thence projects to the level of $A'$, giving $A'''C'$ for the revolved trace upon $D\,C\,B$ of a plane of rays perpendicular thereto. Tangents, as the one through $R$, parallel to $A'''C'$, give the *level* of the points of shade in the meridian plane selected, and, after projection upon $A_1F$, counter-revolve to $D\,C\,B$ as at $e$, whence $e'$.

Fig. 381.

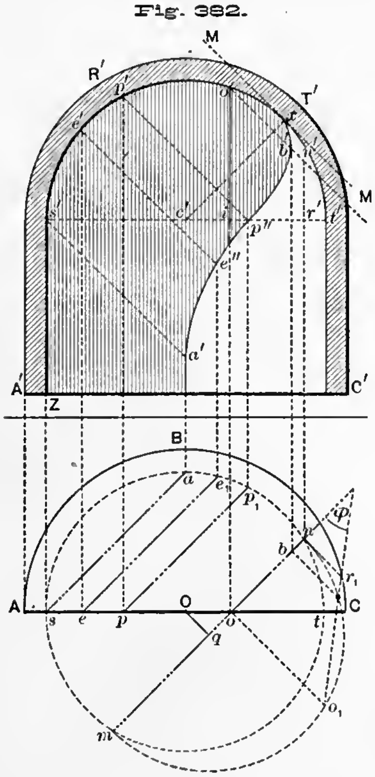



*Points in the meridian profile plane* $L\,X$, as $s''$ and $Z$, are at the same level as those on the apparent contour, owing to the equality of the angles $A_1C\,A$ and $A\,C\,X$.

591. *The shadow on the interior of a niche, cast by its own outlines.* The figure (382) shows the plan $A\,B\,C$, and elevation, $A'R'C'$, of the surface, which may be defined as a vertical semi-cylinder, capped by a quarter sphere.

The direction of the rays being given by $s\,a$, $s'a'$, a series of vertical planes of rays are passed. One of these is $e\,e_1$, containing point $e'$, which casts a shadow, and cutting from the cylinder an element which catches the ray from $e'$ at $e''$.

Fig. 382.




The plane of rays $m\,n$ contains a point $o\,o'$, whose shadow is received by the spherical interior, but which involves no difficulty in its determination, as the ray through $o$ and the circle catching the ray may be shown in their true relation thus: Project $O$ upon $m\,n$ at $q$; then $q\,n$ is the radius of the small circle cut from the top, and $n\,o_1m$ is the revolved position of the circle. Make $n\,r_1$ equal to $n'r'$, and draw $o_1r_1$ for the revolved ray through $o$. It cuts the circle $n\,o_1m$ at a point which counter-revolves to $b$, and thence projects upon $o'n'$ at $b'$.

The tangency of a plane of rays ($M\,M$) perpendicular to V gives the point $x$ at which the shadow begins.

The point $p''$, at which the curve leaves the spherical part, can be exactly determined by a special construction, but is located with sufficient accuracy if enough points have been obtained on either side, by the previous method, for the drawing of a fair curve.

592. *The curve of shade on a warped surface.* This is most readily determined by connecting the points of contact of *tangent planes of rays*, applying the principle that any plane containing an element of a warped surface is at some point a tangent plane to the surface, and if also a *plane of rays* its traces will contain those of any ray intersecting the element.

593. *The shadow of any line upon a warped surface.* Various methods may be employed, among them the following:

(a) Pass a plane of rays so as to cut the line casting the shadow in a point; through the point draw a ray to meet the curve cut from the surface by the plane.

(b) A plane of rays, containing an element on which the shadow falls, will cut the line casting the shadow at the point whose shadow falls on the element selected.

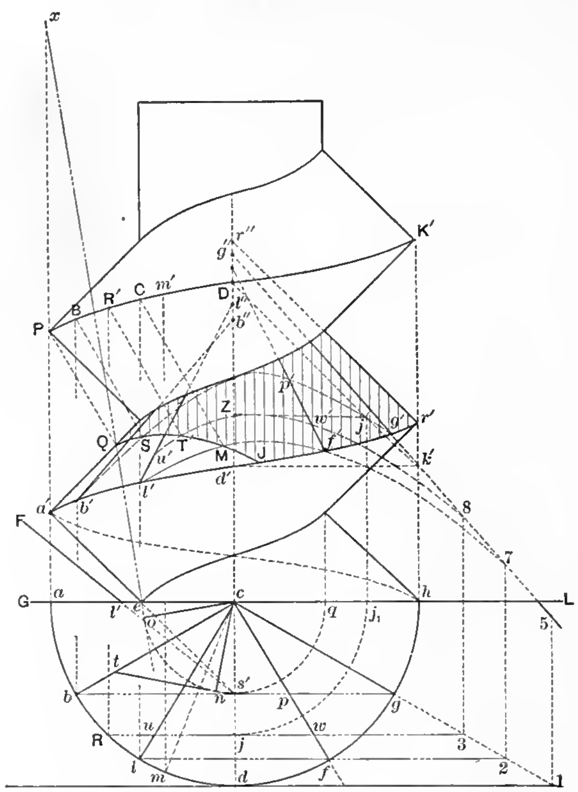
(c) If two lines cast intersecting shadows upon a plane, a ray drawn back from such intersection will meet the line that is nearest the plane in the shadow cast upon it by the point where the same ray meets the more distant line.

594. *The shades and shadows of warped helicoidal surfaces.* Illustrating with the triangular-threaded screw, whose surfaces are helicoidal, let $a'Q$ and $a'e$ (Fig. 383) be the generatrices of the upper and lower surfaces, respectively, the point $a'$ generating the outer helix $a'k'P$, while $e$ and $Q$ generate inner helices of the same pitch as the outer. (Art. 478).

595. *The shadow of the outer helix, $PDK'$, on the surface of the thread below.* Assuming $Rw$, $R'T'$, for the direction of light, a vertical plane of rays $Rjw$ will cut the helix in a point $R$, $R'$, and the helicoid in a curve $u'Z8$, the latter catching the ray from $R'$ in the shadow, $T'$, cast by it. The curve $u'Z8$ is found by projecting $u$, $j$, $w$ and 3, which are the plans of the intersections of the plane with various elements, up to the elevations of the same elements. For the elevation of $j$, which falls on element $cd$, the points $d$ and $j$ are carried to $h$ and $j_1$; $h$ then projects to $k'$, whence $Dk'$ for the revolved elevation of element $cd$. Upon it $j_1$ appears at $j'$, which in counter-revolution returns to $Z$.

Vertical planes of rays parallel to $Rw$ are shown in $ah$, $bg$, $lf$, and with each the process just described is repeated.

Fig. 383.




596. *Points of a curve of shade, by means of a declivity cone.* The direction of light with which we have been dealing so far in the case of this screw puts the entire under-surface of the thread in the shade; but were any portion illuminated, as would occur with light as indicated by $tc$, $t's'$, a curve of shade would have to be determined, point by point, one method for which is as follows: Obtain a tangent plane to the helicoid at some point of any helix. This will be determined by an element and a tangent to the helix. (Art. 478). For the outer helix $a'r'$ the tangent plane at $a'a$ would be $xea'$, as $e$ is the h.t. of the element $a's'$, while $x$ is the h.t. of the tangent at $a'$ (found by making $ax$ equal to the rectified arc $adh$). This plane cuts the axis at $s'$, but is evidently not a plane of rays, since the H-trace, $t$, of the ray through $s'$, does not fall on that of the plane. Since, however, all planes that are tangent to a helicoid at points on the same helix are equally inclined to H, we may find the point of contact of a tangent plane of rays by generating a cone with a line of declivity of the plane just found, passing a plane of rays tangent to said cone, and then finding the parallel plane of rays that is tangent to the helicoidal surface.

The base of the "declivity cone" is $onq$, of radius $oc$, the plan of that line of declivity which lies in a plane with the axis. In $aco$ we see the plan of the constant angle between elements and lines of declivity in the series of planes tangent along the particular helix in question.

The ray through the vertex $s'$ of this cone has $t$ for its trace; $tn$ is therefore the trace of a plane of rays tangent to the cone, and $cn$ the plan of its line of declivity. A parallel plane of rays, tangent to the outer helix, would then contain an element of the helicoid which would be projected as much to the left of $cn$ as $ac$ is to the left of $co$; that is, angle $ncm$ is made equal to $oca$, and $m(m')$ would be that point of the outer helix which belonged to the curve of shade.

A similar process for the inner and any intermediate helices would give points of a curve of shade whose shadow could be found by either of the methods given in Art. 593.

597. When any two surfaces intersect, the shadows cast by either on the other may be found by applying the general principles of Art. 593, care being taken to so avail one's self of known properties of the surfaces as to simplify the construction as much as possible.

# CHAPTER XIV.

DEFINITIONS AND PRINCIPLES.—ARCHITECTURAL PERSPECTIVE FOR EXTERIORS.—PERSPECTIVE OF SHADOWS.—PERSPECTIVE OF INTERIORS BY THE METHOD OF SCALES.

598. A drawing is said to be *in perspective* when its lines correctly represent those of a given object as it would appear from a point of view located at a given finite distance from both it and the plane upon which the drawing is made.

If the representation is not only correct geometrically but is also shaded and colored, it is said to be in *aerial perspective;* otherwise it is simply a *linear perspective.* The construction of the latter is obviously a preliminary to all artistic work in oils or water colors.

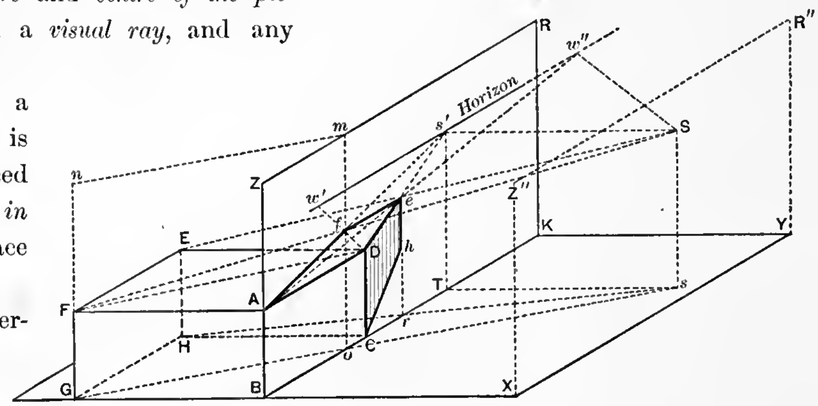
*Perspective plane.* The plane on which the drawing is made is called the *picture plane* or *perspective plane,* and is always understood to be vertical; it will therefore be frequently denoted by the same letter (V) heretofore employed for the vertical plane of projection. It is usually taken between the eye and the object, in order that the perspective may be smaller than the object itself.

599. The general principles and definitions may be illustrated by Fig. 384, which is a pictorial representation of the various elements involved.

*The picture plane* is the vertical surface $BZRK$, later transferred to $XZ''R''Y$.

*Point of sight.—Visual ray.—Visual plane.* $S$ is the supposed position of the eye, and is variously termed *point of sight, perspective centre* and *centre of the picture.* Any line through $S$ is called a *visual ray*, and any plane containing it a *visual plane.*

Fig. 384.




600. $ABCDEG$ (Fig. 384) is a rectangular block whose perspective is to be constructed. It is so placed that one of its faces—$ABCD$—is *in* the perspective plane, making that face its own perspective.

Visual rays, $SF$, $SE$, $SH$, intersect the plane V at points $f$, $e$, $h$, which are the perspectives of the original points. Joined with $A$, $D$ and $C$ they give—with $ADCB$—the perspective of that part of the block which is visible from $S$.

To find the trace, $f$, of ray $SF$, a vertical visual plane may be taken through the ray. $Gs$ is the horizontal trace of such a plane, and $om$ its vertical trace. The ray meets the latter at $f$. Similarly for other rays. Other methods are given in later articles.

The figure illustrates the fact that *the perspective of a vertical line is always vertical;* for the vertical visual plane through $EH$ must cut a vertical plane V in a vertical line, part of which is the perspective $eh$ of the original.

601. *Horizon.* The point of sight, $S$, is projected upon the picture plane at $s'$. A horizontal visual plane will cut the perspective plane in a horizontal line through $s'$, called the *horizon.*

602. *Vanishing points.* The convergence, in a drawing or photograph, of lines representing others known to be parallel on the original object, is a familiar phenomenon. To determine the point of convergence or *vanishing point* of any set of parallels, we have only to obtain the trace on V of a visual ray drawn parallel to the system of lines; for such trace on V may evidently be regarded as exactly covering the point at infinity at which we may conceive the set of parallels as meeting.

The vanishing point of a line is one point of its perspective.

The horizon is the locus of the vanishing points of all horizontal lines.

603. *Vanishing point of perpendiculars.* A *perpendicular* is a line at 90° to V. A visual ray parallel to a perpendicular must obviously be the projecting ray through the eye; and $s'$, therefore, the vanishing point of perpendiculars.

604. *Diagonals and their vanishing points.* Horizontal lines making with V an angle of 45° are called *diagonals.*. $Sw'$ and $Sw''$, the diagonals through the eye, meet the horizon at the *vanishing points of diagonals*, $w'$ and $w''$, also known as *points of distance*, since they are as far from the *projection* of the eye as the space-position of the latter is from V.

605. *Lines parallel to the perspective plane* have their vanishing points at infinity; or, in other words, the lines and their perspective representations are parallel. This is illustrated by $EH$ and $eh$ in Fig. 384.

606. *Perspective by trace and vanishing point.* Since any point in the perspective plane is its own perspective, we may obtain the indefinite perspective of any *line*, as $FA$, by joining $A$—its trace on V—with its vanishing point, the latter being $s'$ in this case, as the line mentioned is a perpendicular. The visual ray $SF$ then intersects the indefinite perspective $As'$ at $f$, when $Af$ is readily seen to be the definite perspective of $AF$. (For application see Art. 612).

607. *Perspective by diagonals and perpendiculars.* The student has already discovered, without formal statement of the principle, that a point is determined in perspective by its being the intersection of the perspective of two lines passing through the original point. We saw in the last article that the perspective of $F$ was obtained as the intersection of a ray $SF$ with a line $As'$, which might be regarded either as the trace of a visual plane or as found by joining trace of line with vanishing point. Obviously any pair of lines may be drawn through a point, and the intersection of their perspectives noted; but the auxiliaries which, on account of their convenience, are most frequently used in perspectives of interiors, are the diagonals and perpendiculars already defined.

In the figure, $FD$ is the diagonal of the square top of the block; $Sw'$ is parallel to $FD$, and $w'$ therefore the vanishing point of diagonals, to use in getting the perspective ($Dw'$) of said diagonal. This intersects $As'$ (perspective of perpendicular $AF$) at $f$.

608. Having illustrated pictorially the principles most employed in linear perspective, we have next to show how they are applied to the orthographic projections which are usually all that the draughtsman has, with which to start his constructions. Obviously, the perspective plane cannot be rotated backward into coincidence with the paper on which the object is represented in plan, without the latter drawing being in most cases overlapped by the perspective representation. The usual—probably because the most natural—way to avoid this difficulty, is to imagine the plane V transferred forward to some position $Z''XYR''$, where, if rotated into the paper about its trace $XY$, the perspective will clear the auxiliary views.

609. In Fig. 385 the method just described is illustrated as applied to the block of Fig. 384, and both figures may be referred to in the following description:

$BK$ is the trace (and orthographic representation) of the entire perspective plane $BZRK$, and $XY$ its transferred position, upon which the elevation of the object is drawn ($A'B'C'D'$). The plan $ADEF$ is drawn back of $PK$, and in the same relation to it as the object to V. $Ts$ is the same in each figure.

Fig. 385.

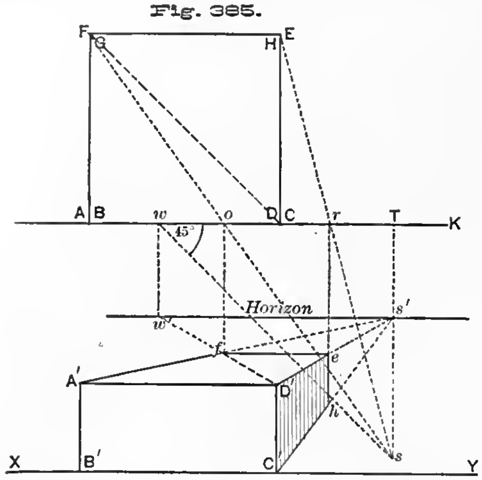
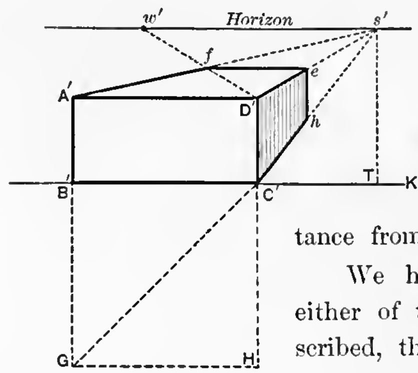



Fig. 386.




The *horizon* is located in relation to the ground line $XY$ in Fig. 385, at a distance from it equal to $Ts'$ (or $Ss$) in Fig. 384.

*The vanishing point of diagonals*, $w'$, is at a distance from $s'$ equal to $sT$.

We have now merely to apply either of the methods previously described, thus:

(a) $Es$ ($D's'$) is the visual ray through $E$ (the latter being projected on V at $D'$). $Erh$ is the vertical visual plane through this ray; $rh$ is its vertical trace, and $e$ is the trace of the visual ray and therefore the perspective sought.

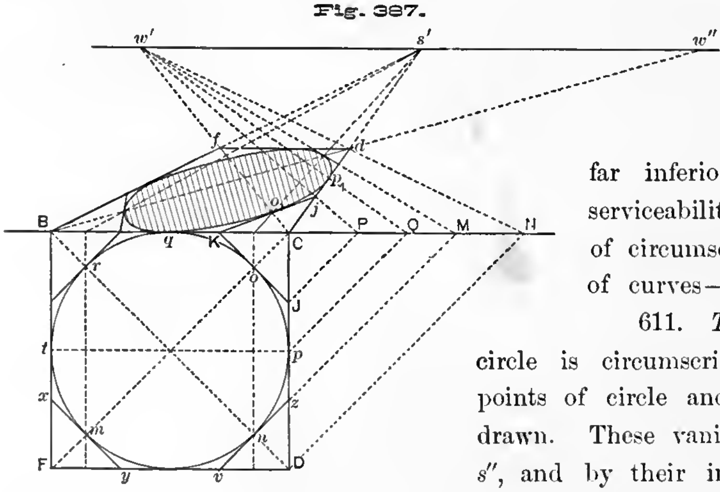
(b) $D'$ is the trace of $ED$, and $D's'$ is its indefinite perspective, drawn to the vanishing point of perpendiculars. Ray $sE$ meets the trace $Bk$ at $r$, whence $e$ for the perspective desired.

(c) Wanting the perspective of $F$, we may draw through it the perpendicular $FA$ and the diagonal $FD$. These, being on the top of the block, meet V at the level $A'D'$. $A's'$ is therefore the perspective of the perpendicular, $D'w'$ that of the diagonal, and their intersection $f$ the point desired.

610. *The method by inverted plan.* In Fig. 386 the same perspective as in the preceding figures is obtained by assuming that the object has been rotated 180° about $BK$, so that it appears, inverted, in front of the perspective plane. Diagonals and perpendiculars then give the same result as before.

Any diagonal, as $GC'$, being inverted, is drawn in perspective in its true direction, $D'w'$.

Fig. 387.



This method is quite convenient when dealing with plane figures; but for large and complicated objects the conception of inversion is confusing, and renders it far inferior to the other. To show, however, its serviceability in the field indicated, as also the device of circumscribing polygons—usually resorted to in case of curves—Fig. 387 is given.

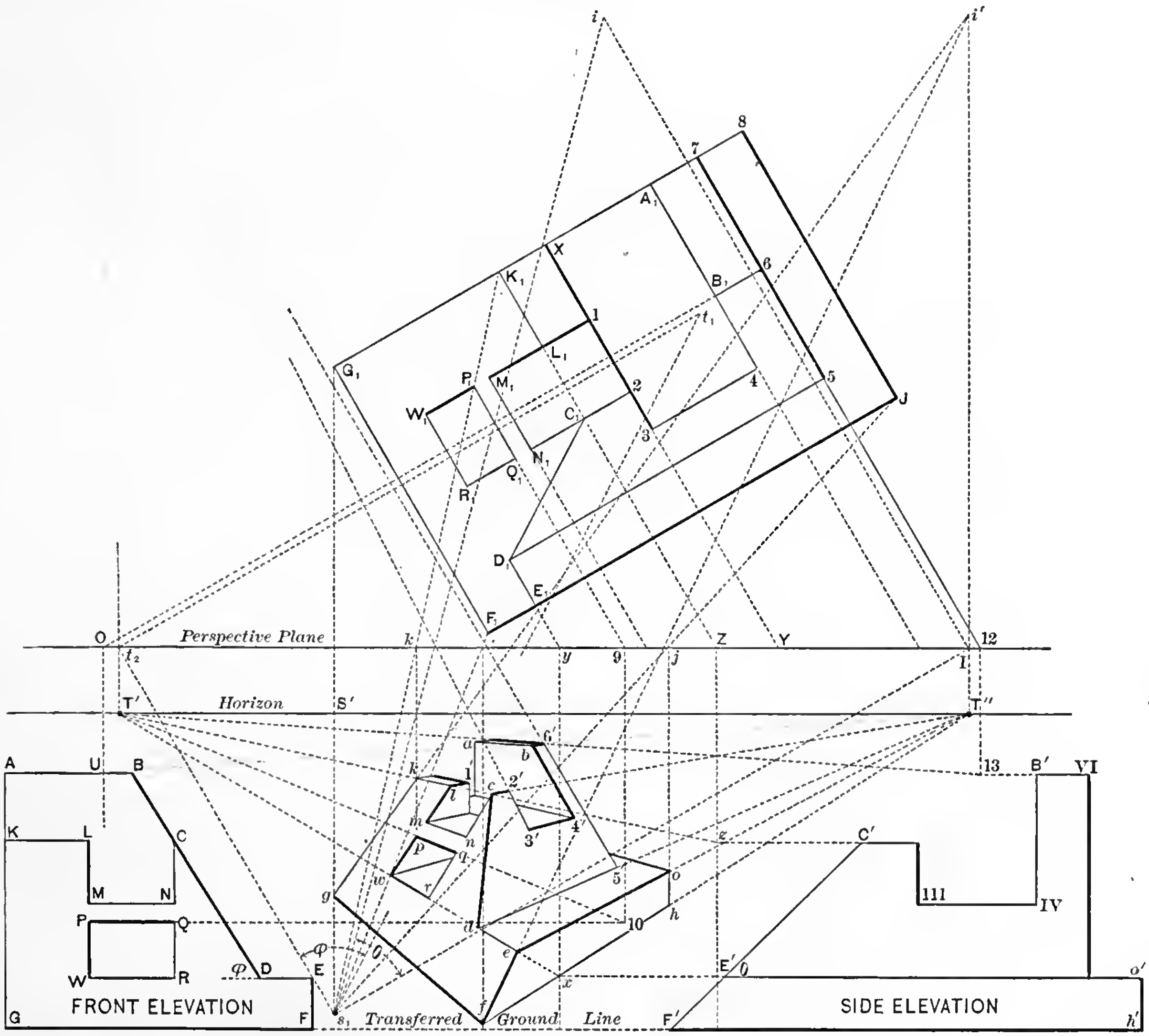
611. *The perspective of a circle.* In Fig. 387 the circle is circumscribed by an octagon. Through the various points of circle and octagon diagonals and perpendiculars are drawn. These vanish, in perspective, at the points $w'$, $w''$ and $s''$, and by their intersections either give points of the circle directly, or lines to which the perspective of the circle can be sketched in tangentially.

The perspective of a circle will be a circle only when it is parallel to V, or when the visual cone to its points is cut by V in a sub-contrary section. In all other cases it is an ellipse, when the circle is on the opposite side of V from the eye.

612. *Perspective by Trace and Vanishing Point, with special reference to its application to architectural constructions (exteriors).*

In further illustration of the method of Case (b) of Art. 609, which is generally used in drawing the perspective of the exteriors of residences and other architectural constructions, Fig. 388 is presented. The object dealt with has not only horizontal and vertical lines but also edges inclined at

Fig. 388.




various angles to a horizontal plane, so as to illustrate the method of dealing with any direction of line that can occur in a house perspective. The only reason for not presenting the plans and elevations of some actual building; is the impossibility of reducing such a series of views to the necessary limits of our illustration without sacrificing clearness as to the constructions made therewith; but if for the given elevations and plan there were substituted the elevations of a house, together with a general roof and wall plan of that part of the house which is visible from the point of view selected, the procedure from that point would be identical with that shown above.

The central view gives the best idea of what the object is like, a block devised simply with reference to compact illustration of the various principles involved.

*Lines of height.* Vertical planes through the edges of the plan, as $D_1 y$, $K_1 Z$, $XY$, are first drawn, and their traces shown on the (transferred) perspective plane in the verticals $yx$, 9–10, $Zz$, etc. These, in architect's parlance, are *lines of height*, since upon each is laid off the true height of lines in the plane it represents. These heights are most conveniently located by projecting over directly from one or the other of the elevations. Thus, the height of $K_1 C_1$ is seen at $C$ on the front elevation, and at $C'$ on the other. Projecting from the latter, we have $z$ as the height at which $K_1 L_1$ would meet the perspective plane; its *trace*, therefore, to use as in Art. 606. Similarly, $6B_1$ produced gives $OU$ for its line of height, and $BA$ cuts it at the trace $U$.

$P_1 Q_1$ gives the trace 9–10, upon which $PQ$ projects, giving 10 for the height of both $P$ and $Q$.

613. *The vanishing points.* With $s_1$ as the plan of the eye in its relation to the original position of V, draw the horizontal line $s_1 I$ parallel to the longer lines of the plan, and project $I$ to $T'''$ on the horizon, for the vanishing point of that set. Similarly draw $s_1 t_2$, parallel to those horizontal lines of the object that are perpendicular to the first set, getting $T'$ for the other vanishing point of horizontal lines. Were there horizontal lines in other directions on the object, their vanishing points would have to be determined by an analogous process.

*The vanishing point of lines making angle* $\theta$ (see side elevation) *with the horizontal* is at $i'$, found thus: Draw $s_1 I$ parallel to the plans of the lines whose inclination is $\theta$. At $I$ draw a perpendicular to $s_1 I$ and prolong it to meet, at $i$, a line $s_1 i$ making $\theta$ (at $s_1$) with $s_1 I$. Make $T'''i'$ equal to $Ii$, when $i'$ is recognized as the trace of a visual ray parallel to the lines whose inclination was given; for when the triangle $s_1 i I$ is rotated upon its base $s_1 I$ until vertical, and then placed with said base at the level of the eye, we would evidently find $i$ at $i'$.

*The vanishing point of lines inclined* $\phi°$ *to* H is found by duplicating the last procedure in every detail, $s_1 t_2 t_1$ being then the triangle whose altitude $t_2 t_1$ is laid off vertically from $T'$, and toward which $3'2'$ and $5'6'$ converge.

The remaining construction is as follows: Vertical visual planes are drawn through $s_1$ and all points of the object. To avoid complicating the lines only a few of these are shown, $s_1 G_1$, $s_1 P_1$, $s_1 F_1$, $s_1 J$. Taking $K_1 P_1 s_1$ as illustrative of all, we draw its vertical trace $kkp$. The line of heights for the point $P$ being 9-10, project $P$ upon the latter at 10; then 10-T$'$ is the perspective of 9-$P_1$, and its intersection $p$ with the vertical $kk$ is the perspective sought. $K_1$, in the same visual plane, has its height projected from $K$ to $z$, upon the line of heights through $Z$; then $zT'$ gives $k$. The perspectives of all the other points might be similarly found; but with two or three points thus obtained we may find the various edges by means of the vanishing points, thus: Starting with $e$, for example, prolong $i'e$ to meet at $f$ the trace of visual plane $s_1 F_1$; then $fT'$, stopping at $g$ on trace $s_1 G_1$. As $gk$ and $fe$ have the same vanishing point, we find $k$ as the intersection of $gi'$ and $zT'$. Then $T'k$ prolonged gives $l$ and $c$ on traces of visual planes (not drawn) through $L_1$ and $C_1$.

The point $d$ being found independently, we join it with $c$ for the edge $cd$, for which we might also find a vanishing point thus: Obtain the base angle of a right triangle of base $C_1 D_1$, and altitude equal to height of $C$ above $R$; then use this angle (which we may call $\beta$) and the direction $D_1 C_1$ exactly as $\theta$ and $F_1 J$ were used in the construction giving vanishing point $i'$.

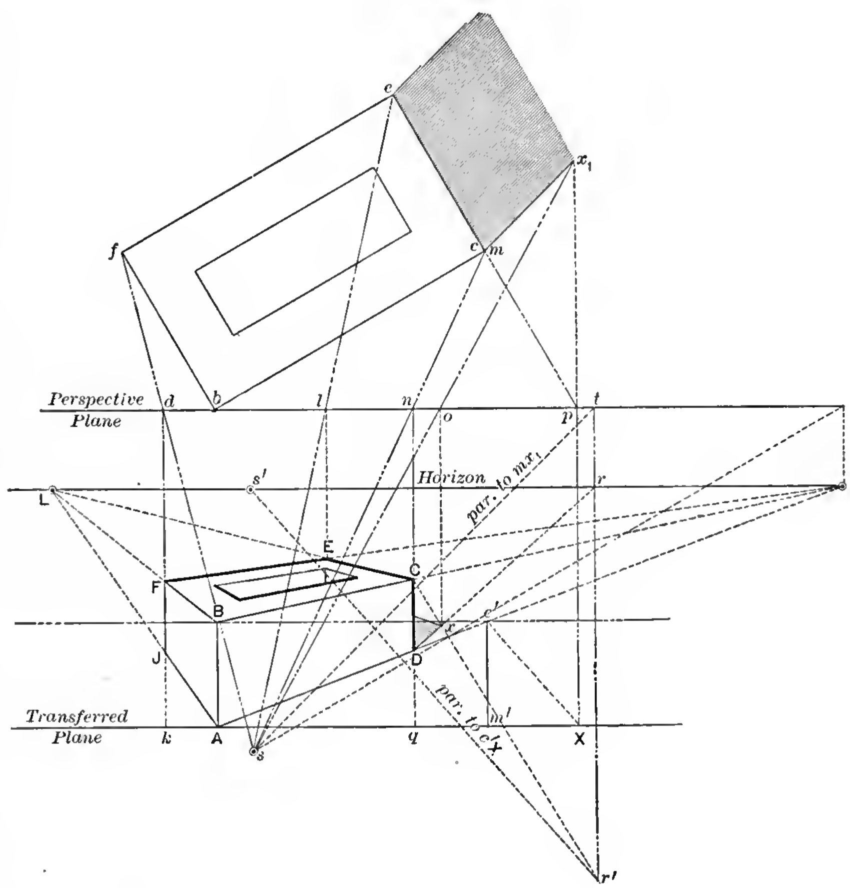
614. *Perspective of Shadows.* These might be obtained from their orthographic projections in every case, but usually a shorter method is employed. Both ways are illustrated in Fig. 389.

The object whose perspective and shadows are to be constructed is a hollow rectangular block,

whose plan is $fbcd$, and whose height is seen at $AB$. The corner $b$ being in the perspective plane, we have in $AB$ the perspective of the front edge.

The vanishing points $R$ and $L$ having been found from $s$, as in the last problem, draw $AR$ and $BR$, and terminate them on the trace $nq$ of the visual plane $sm$. Similarly, terminate $AL$ and $BL$ upon the trace $dk$ of the vertical visual plane $sf$. Then $FR$ and $CL$ give the rear corner $E$, etc.

Fig. 389.




*The shadow.* Let $c'm'$ be the orthographic elevation of the edge whose plan is $c$. Then if a ray of light through $c$ ($c'$) has the projections $cx_1$, $c'X$, we shall have $x_1$ for the shadow of $cc'$. In the same way the shadow might be completed in orthographic projection, a portion only, being, however, actually indicated. Then, treating $x_1$ like any other point whose perspective is desired, we would find $r$—the vanishing point of horizontal lines parallel to $mx_1$, and draw $Dr$ for the perspective of the plan of a ray; then $x$, the intersection of $Dr$ with the trace $ox$ of the vertical visual plane $sx_1$, is the perspective of the shadow of $C$.

$CE$ being horizontal, its shadow on H is in reality parallel to it, and, perspectively, has the same vanishing point; hence draw from $x$ toward $L$ to complete the visible portion of the shadow.

$CD$ being a vertical line, has its shadow $Dx$ in the direction of the projection of rays on H.

615. *The perspectives of shadows, without preliminary construction of their orthographic projections,* are thus obtained: In Fig. 389, with $c'X$ and $cx_1$ as the orthographic projections of a ray, draw $s'r'$, $st$, for the parallel visual ray, when $r'$ is seen to be *the vanishing point of rays.* Then $r$ is obviously the *vanishing point of horizontal projections of rays;* and for shadows on horizontal planes the two points thus found are sufficient. For the shadow of $C$ we have merely to take the direct ray $Cr'$, and the plan $Dr$ of the same ray, and note their intersection, $x$.

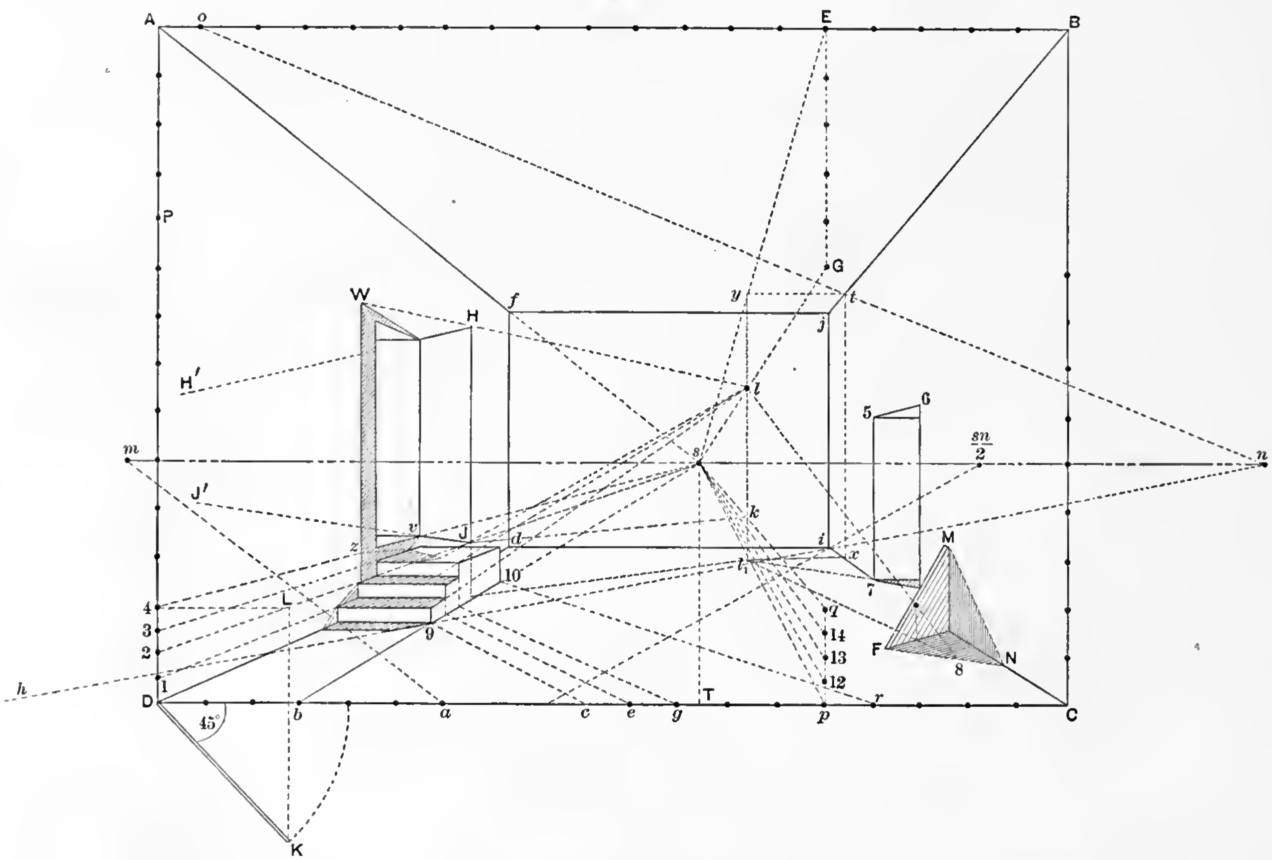
For shadows on a set of parallel planes that are not horizontal, we would replace $r$ by the vanishing point of *projections of rays* on the planes in question.

616. *Perspective by the method of scales.* (a) In Fig. 390 let $s$ be the point of sight, $mn$ the horizon, and $m$ and $n$ vanishing points of diagonals. Attention is called again, by way of review, to the fact that the real position of the eye in front of the perspective plane is shown by either $ms$ or $ns$, and that all horizontal lines inclined 45° to V will converge to $m$ or $n$.

We now apply the properties of the 45°-triangle thus: To cut off any distance, perspectively, upon a perpendicular to V, lay off the same distance parallel to V and draw a diagonal. If the room is to be twenty-two feet deep, make $Ch$ equal to that number of units and draw the diagonal $hn$, cutting off the perpendicular from $C$ at $i$, making $Ci$ the perspective of the given depth. The rectangle $ABCD$ having been laid off *in* the perspective plane, from given dimensions and to the same scale, draw from $A$, $D$ and $B$ toward $s$, terminating these perpendiculars on a rectangle obtained by drawing $id$ and $ij$, then $jf$ and $df$ parallel to the corresponding sides of the larger rectangle.

(b) *Reduced vanishing points.* In case the point of sight has been taken at such a distance from the perspective plane as to throw $m$ and $n$ beyond convenient working limits, we may get the same

Fig. 390.




result by bisecting or trisecting $sn$ and taking the same proportion of the distance to be laid off. Thus the point $i$ might be obtained by bisecting $Ch$ and drawing a line to the middle of $sn$. We might equally well lay off from $C$ toward $h$ any other fraction of $Ch$, and draw thence to a point on the horizon whose distance from $s$ was the same fraction of $sn$.

(c) *The perspective of the steps.* Let it be required to draw a flight of three steps leading to a doorway in the left wall. If the lowest step is to be six feet from the front of the room, make $Da$ equal six feet and draw $am$, cutting $Dd$ at a corner of the step in question.

If the steps are to be three feet wide, make $Db$ equal to three units, and draw $bs$ for the trace of the vertical plane of the sides of the steps. Making $ac$ three feet, draw $cm$, getting point 9, which should be even with the first corner found.

The widths of the steps being laid off from $c$ at $e$ and $g$, and their heights at 1, 2, 3 on $DA$, their perspective is completed by a process which should need no further description.

(d) *The doorway in the left wall.* Assuming this to be the same width as the landing, which—as seen at $gr$—is evidently four feet; and also that the walls of the hallway are in the planes of the front and back of the landing, draw vertical lines from the left-hand corners of the latter, terminating them by a perpendicular $Ps$ drawn from a point $P$ whose height (ten units) is that of the top of the doorway. 3-4 shows the height of step from landing to hallway.

*The perspective of the door* is obtained in this case on the supposition that it is open at an angle of 54°, for which a vanishing point (not shown) lies on $sm$ prolonged, and from which a line $J'v$ gives the direction $vJ$, and similarly $H'H$ for the top of the door.

To find $J$ we may draw $DK$ at 54° to a *vertical* line (the 45°-angle indicated is an error, should be 36°) so as to represent a four-foot door swung through the proper arc, when by projecting up $DK$ to 4-$L$, which is the level of the bottom of the door, a perpendicular $Ls$ will cut $J'v$ at $J$. Then a vertical line from $J$ will cut the line $H'$ at $H$.

Were the door actually open 45°, the edge $Jv$ would pass through $m$.

The hallway on the right has its corner 7 at a distance of thirteen feet from $C$, and is seven feet high. The width of the passage may be ascertained by the student.

*The method of getting the perspective of a door by means of an auxiliary circle* is shown in Fig. 391.

(e) *The location of the light, $l$.* To locate the light five feet from the right wall, move five units from $B$, to $E$, when $Es$ will be the trace, on the ceiling, of the vertical plane containing the light.

If the light is to be five feet below the ceiling, mark off five units down from $E$, when $Gs$ will be a horizontal line giving the *level* of $l$.

Finally, to have the light a definite distance back, say eighteen feet, make $Bo$ eighteen units on the front edge of the ceiling; draw $on$ and get $t$, when $txl_1y$ will be a plane at the required depth, and its intersection $l$ with $Gs$ will be the position of the light.

(f) *The shadows.* As in any other shadow construction, we have to note, in any case, where a direct ray through a point meets the projection of the same ray. All *horizontal projections of rays* will pass through $l_1$, which is the projection of the light on the floor.

For the triangular block $FM$, we take a direct ray $l8$, through any point of the edge casting the shadow, and $l_18$ for the projection of the same ray; then 8 is the shadow of the point selected. At $F$, where the edge meets the floor, the shadow begins, hence $F$-8 is the *direction* and $FN$ the *extent* of the shadow cast on the floor, and, obviously, $NM$ that received by the side wall.

(g) *The shadow of the door.* $JHll_1$ is a vertical plane of rays containing $JH$. It cuts the left wall in a line $Wz$, found by continuing the trace from $l_1$ to meet $Dd$, erecting therefrom a vertical line and cutting it at $W$ by a ray from $l$ through $H$.

The shadow of $Jv$ on the landing has the same vanishing point as $Jv$. When it meets the side wall it joins with $v$, since there the line casting the shadow meets the surface receiving it.

The shadow of $J$ is at the intersection of ray $lJ$ with the trace (not drawn) of the plane of rays $JHl$ upon the top of the block. This would be found thus: Where the h. t. of said plane cuts the edge 9-10 draw a vertical line, and from the intersection of the latter with the top edge of the landing draw a line to the point below $l$ on the line 14-$s$. This will give the *direction* of the shadow of $HJ$ on the landing, since the line 14-$s$ is at the level of the top of the landing.

(h) *The shadows of the steps.* The plane $yEGl$ has the trace $l_1p$ on the floor; and if on the vertical $pq$ we lay off distances equal to the heights of the steps and draw vanishing lines to $s$, these will cut $ll_1$ at points which may be regarded as the projections of the light upon the planes

of the tops of the steps, and should be used in getting the directions of the shadows of vertical lines upon said tops. The direction of the shadow of the vertical edge at 9 is given by $l_1 9$, which is made definite as to length by a ray from $l$ through its upper extremity. The shadow on the floor is then parallel to the front edge till it meets the side wall, where it joins with the end of the line casting the shadow.

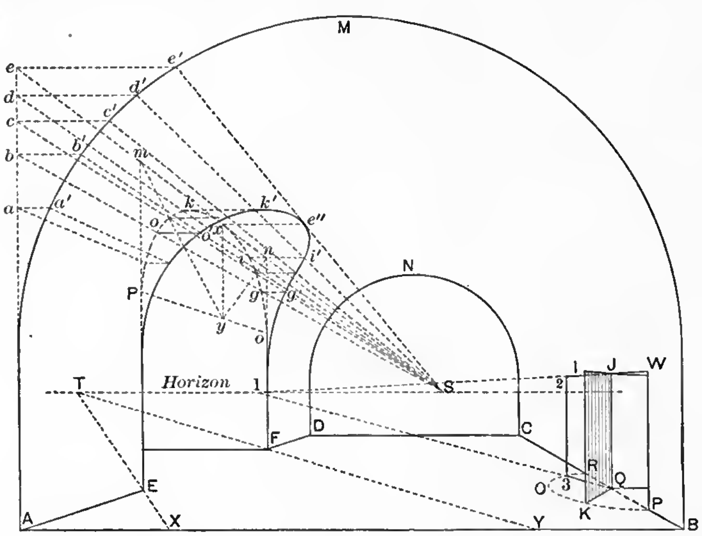
The vertical edges of the second and third steps would cast shadows whose directions would be found by means of the points above $l_1$ on 12-*s* and 13-*s*, and which would run obliquely across the tops, instead of covering them entirely, as shown, incorrectly, by the engraver.

617. *The perspective of a right lunette, the intersection of two semi-cylindrical arches of unequal heights.* Let $AMB$ be the front of one of the arches and $DNC$ the opposite end, at a distance back which may be found by drawing from the vanishing point of diagonals, $T$, a line $TD$ to meet $AB$ produced, giving a point whose distance from $A$ is that sought.

Let the smaller passage be at a distance $AX$ back of $A$, and equal to $XY$ in width.

Continue the vertical plane on $AD$ to the level $eS$ of the highest element of the smaller arch, and in that plane construct $Pmno$—the perspective of half of a square whose sides equal $XY$. In this draw the perspective of a semicircle $Pkgo$. At $ee'$, $dd'$, etc., we see the amounts by which the elements of the side cylinder extend past the plane $AenF$ to their intersection with the main arch, and these in perspective are ordinates of the curve $o'e''g'$. For any one, as $bb'$, draw $bS$, cutting the semicircle $Pko$ at $o$ and $g$. Horizontals through these points will be those elements of the smaller cylinder that lie in the horizontal plane $bb'S$; and the perpendicular $b'S$ cuts them at the points $o'$ and $g'$ of the intersection.

Fig. 391.




618. *The perspective of a door, found by means of an auxiliary circle in perspective.* Let $QP$, Fig. 391, be, perspectively, the width of the given door. Construct $PKO$, the perspective of the circle that $P$ would describe as the door opened. If $QP$ were to swing to $Q3$, the prolongation of the latter would give 1 on the horizon for its vanishing point, which joins with $J$ for the *direction* of the top edge, the latter being then limited at 2 by a vertical through 3. Similarly, $KQ$ prolonged to the horizon, gives a vanishing point from which a line through $J$ gives the top edge $JI$.

619. *The perspective of a groined arch.* If the axes of two *equal* cylinders intersect, the cylinders themselves will intersect in plane curves, *ellipses.* In the case of arches intersecting under these conditions the curves are called *groins*, and the arches *groined arches*, when that part of each cylinder is constructed which is exterior to the other, as in Figs. 392 and 393.

Were $G$ joined with $M$ in Fig. 392 and the line then moved up on the groin curves $Go$ and $Mg$ to $h$, it would generate *between those curves* one-quarter of the surface of a *cloistered arch*, but the curves would still be called groins.

Fig. 392 represents in oblique projection that portion of the structure which is seen in perspective above the pillars in Fig. 393.

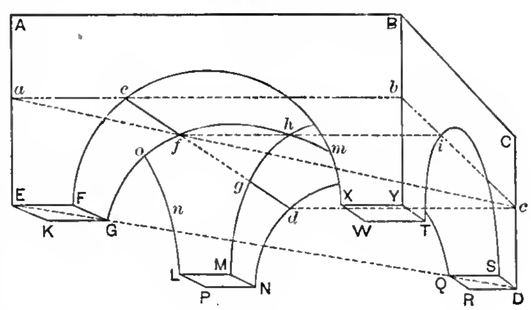
The pillars are supposed to be square, and to stand at the corners of a square floor set with square tiles.

Each pillar rests on a square pedestal and is capped by an abacus of the same size except as to thickness.

Taking the perspective plane coincident with the faces of the pedestals and abaci, make $a\,b\,l\,t$, Fig. 393, of any assumed size; prolong $b\,l$ until $l\,v$ equals the height assigned to the pillar; then complete the front of the abacus on $q\,v$ as an edge, to given data.

Locate on $t\,l$ a point whose distance from $l$ equals that of the front face of pillar from the perspective plane, and draw therefrom the perpendicular $h\,G\,s$; also draw the diagonal $l\,h\,d$, giving $h$ for a starting corner on the base of pillar. A parallel to $t\,l$ through $h$ is cut by the diagonal $t\,E\,d$ at $E$. The same diagonal gives $G$ and two points on the diagonally-opposite pillar, corresponding to $E$ and $G$.

Fig. 392.




The vertical line through $h$ is cut by a diagonal from the $v$-corner of the abacus at the point where that edge meets the abacus, and the completion of the perspective of the top is identical with that just described for its base. The prolongation of $h\,v$ meets a diagonal from $q$ at the point where the front semicircle begins on the top of the abacus. Joining it with the corresponding point on the other abacus and bisecting such line gives the centre of the front curve, which may be drawn with the compasses, as the circle is parallel to the perspective plane. Similarly for the back semicircle.

*The perspectives of the groins and side semicircles.* As the cylinders on which these curves lie are either parallel to or perpendicular to the paper we may refer to them as the parallel and perpendicular cylinders, respectively. The elements of the latter will converge in perspective to the point of sight, as $m\,c$ and $y\,L\,c$, Fig. 393. On the other cylinder they will be parallel in perspective.

A horizontal plane of section, as that through $a\,b$ (Fig. 392) or $B\,T$ (Fig. 393) will cut a square $a\,b\,c\,d$ from the outside of the structure, and two elements from each cylinder. Either diagonal of this square, as $a\,c$ (Fig. 392) will cut the elements in points of the groin. In Fig. 393 the diagonal $B\,d$ cuts the elements $m\,s$ and $y\,s$ at $e$ and $e$, two points of the groin. These points also belong to elements of the parallel cylinder, and the latter, if drawn, will meet the side walls in points of the side semicircles. This is shown in Fig. 392 by drawing the element $f\,i$ to meet the trace $b\,c$ at $i$. In perspective this is seen in the horizontal element through $e$ (Fig. 393) which meets the perspective perpendicular $B\,s$ at point $f$ of the side curve.

A number of planes should be treated like $B\,T$ to give enough points for the accurate drawing of the curves.

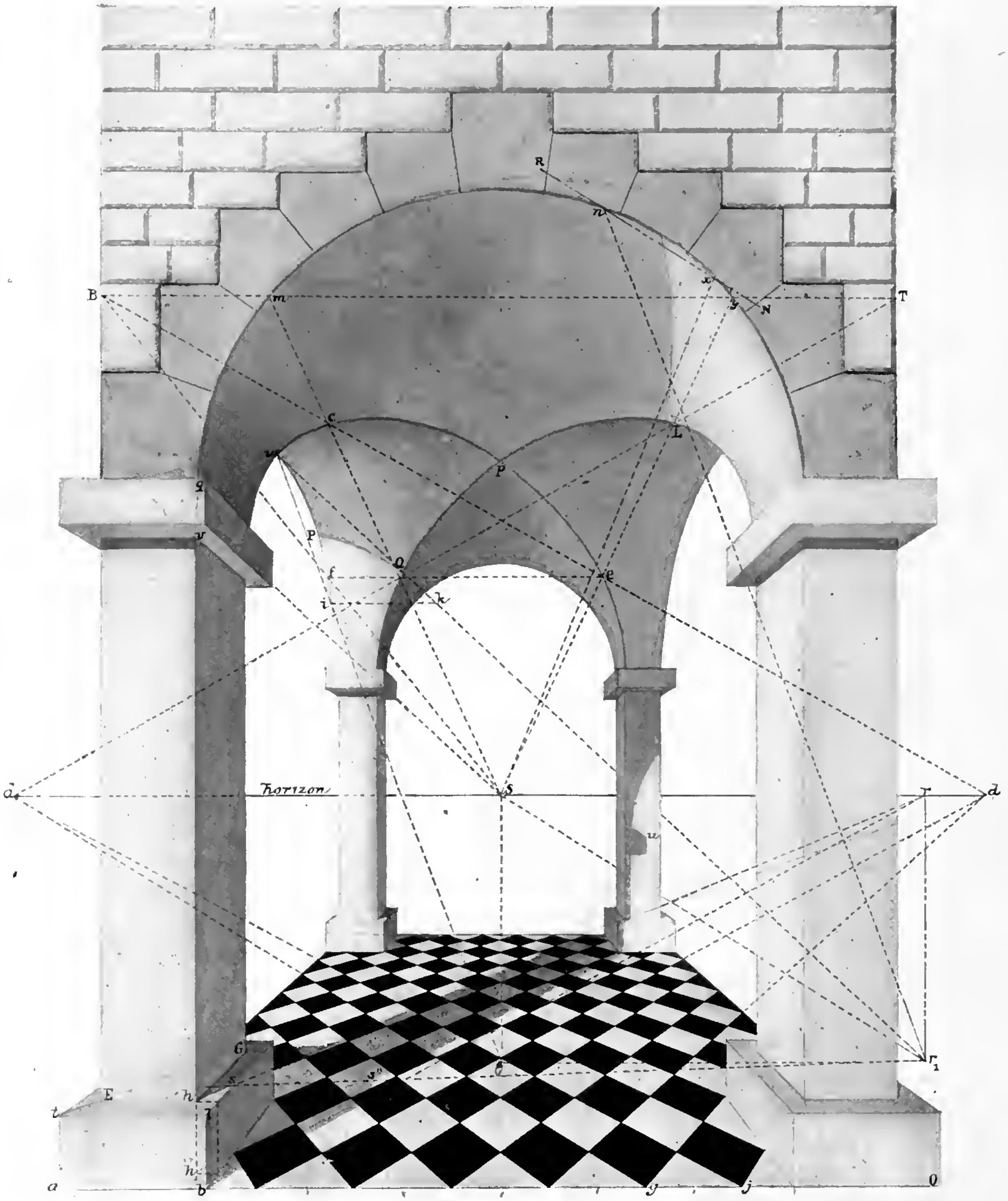
*The tiled floor* is made of squares whose diagonal is seen in true size at $g\,j$. By laying off the latter on $a\,O$ and drawing diagonals to the vanishing points $d$, the floor is rapidly laid out.

*The shadow of the left-hand front pillar and of its abacus.* Assume $r_1$ and $r$ for the vanishing point of *rays* and of their *horizontal projections*, respectively, remembering that these points must be on the same vertical line, since a ray and its plan determine a *vertical* plane, which can intersect another vertical plane only in a vertical line.

The ray from $l$ to $r_1$ meets its projection $b\,r$ at the shadow of $l$ on the floor, whence a perpendicular to $s$ would give the direction of the shadow of $l\,s$. Join $h$ with $r$; it runs off the pedestal at $s$, whence a ray to $r_1$ will give the shadow $s''$ on the floor, from which $s''r$ is one boundary of

the shadow of the pillar. This meets the further pedestal, upon whose front the shadow runs up vertically, being the shadow of a vertical line. On the top of the pedestal the shadow continues

Fig. 393.



toward $r$ till it meets the face of the pillar, where it again runs vertically until merged in the shadow of the abacus $q\,v$.

*The shadow of the abacus $q\,v$* on the diagonally-opposite pillar consists of a horizontal shadow cast by a small portion of the lower front edge; a vertical portion cast by $q\,v$; and a portion running

from $u$ parallel to $s r_1$ and cast by a part of the edge running from $q$ toward $s$. The plane of rays through the edge last mentioned would be perpendicular to the paper, and its trace necessarily parallel to that of the projecting plane of the ray of light through the eye.

*The front semicircle casts a shadow on the rear pillar,* whose centre is found by extending the plane of the front of the pillar sufficiently to catch a ray drawn through the centre of the original curve. The radius of the shadow would be the (imaginary) shadow of the radius of the front semicircle.

*The shadow of the arch-curve on the interior of the perpendicular cylinder* is found thus: Pass planes *of rays* parallel to the axis of the cylinder. $R N$ is the trace of one such plane. It is parallel to $s r_1$. It cuts the element $x S$ from the cylinder, and the point $n$ from the curve casting the shadow. The ray $n r_1$ meets the element $x S$ in a point of the curve of shadow. The shadow begins at the point where a plane of rays, parallel to $R N$, is tangent to the face curve of the arch.

*The shadow,* $P Q$, *of the side semicircle,* is found by taking planes of rays parallel to the axis of the parallel cylinder. The traces of such planes upon the side face of the structure, (which is perpendicular to V), involve the location of the vanishing point of projections of rays on profile planes.

The ray of light through the eye (which meets the perspective plane at $r_1$), projected on the profile plane through the eye, appears at $S o$; hence $o$, at the level of $r_1$) and on the vertical line through $s$, is the vanishing point sought. Lines radiating from $o$, as $o w$, are perspective traces of planes of rays. Each cuts the arch curve in a point as $w$, and the cylinder in an element as $i k$. The ray $w r_1$ through the point then meets the element in a point of the shadow. As $k$ is not on the real part of the cylinder, it is useful only in connection with other points similarly found, to determine the shape of the curve $P Q$.

620. *Some hints as to planning a drawing, and on intelligent criticism of works of art.*

At this point, though we grant acquaintance on the part of the student with the principles of this and the preceding chapter, upon whose correct application the success of the architect or artist so largely depends, there is no certainty that he could make a drawing which should not only be mathematically correct but also pleasing to the eye, or that he could pass just criticism on the work of others. The artistic sense, to be cultivated, must be innate; and originality or inventiveness can only in small degree be inculcated by either precept or example. Yet one who is neither the "born artist" or "natural architect" can, by the mastery of a few cardinal principles, be not only guarded against the making of glaring errors, but also have his interest in works of art materially enhanced. In bringing this chapter to a close it seems advisable, therefore, to give a few hints with regard to the more important points upon which successful work depends.

In the first place, the location of the point of view is by no means immaterial. It is not well to attempt to include too much in the angle subtended by the visual rays to the extreme outlines of the object drawn, and the frequently-recommended angle of 60° may safely be taken, as, in most cases, the maximum for pleasing effect. Nor should the eye be taken too near the perspective plane, since this involves a degree of convergence amounting to positive distortion, as all must have noticed in photographs of architectural subjects taken at too short range. In case of an error in first location of view-point, one can diminish the convergence of the lines without reduction in the size of the perspective result, by a simultaneous increase of the distance of the eye from the perspective plane and of the latter from the object.

Usually, and, in particular, in an architectural perspective, the eye should not be opposite the centre of the structure, a more agreeable effect resulting from a lack of rigid geometrical symmetry and balance. Nor should the lines of the structure make equal angles with the perspective plane.

When viewing a picture, the endeavor should be made to put one's self at the point of sight selected by the artist. In fact, one will instinctively make the attempt so to do; but, to succeed, it is well to bear in mind the principles previously set forth as to the location of vanishing points.

Among other essential preliminaries to which careful thought must be given is the quality called "*Balance*" by artists. It is the adjustment of the various elements of a picture, so that while leaving no doubt as to its purpose there shall not be over-emphasis of its main feature, but a general interest maintained in the various accessories, the office of the latter to be, evidently, however, contributory to the central idea or object. Probably no better example of balance can be found, to say nothing of its illustration of the other requisites of a good picture, than Hofman's well-known painting of Christ preaching on the shore of Galilee. (National Gallery, Berlin).

When a drawing has been well planned as to its geometrical character and arrangement, and its main lines pencilled in, the next point to be considered is the style of finish. For architectural work there may be all degrees, from the barest outline drawing in black and white, to the most highly finished water color work, with sky effects, and foliage, water and figure "incident."

To secure the sketchy effect which is so desirable, and avoid the harsh exactness of geometrical diagrams, all inked lines should be drawn free-hand except in work upon which considerable free-hand shading is intended. And even when the inked lines are ruled, they may preferably be in a succession of dashes of varying length, rather than in continuous lines.

Whatever guide lines are required for the boundaries of surfaces that are to be "rendered" (i. e., brush-tinted) in water colors, should be ruled in pencil only.

Probably the most practical as well as pleasing style of work is that in which each stroke of pen or brush suggests, by its location or its weight, quite as much as it actually represents,—*impressionist* work, in technical language.

Scarcely second to correct planning and outlining is the *chiaroscuro*, or light and shade effect due to the *values* or intensities of the tones given to the various surfaces. Not exactly synonymous with it, yet dependent upon it, is the quality termed *atmosphere*, upon which the effect of *distance* largely depends. When well rendered, the foreground, background and middle distance are harmoniously treated, and the idea conveyed is the same as by a view in nature, changing from the clearness and sharp definition of that which is nearest, to the hazy air and general indistinctness of detail of the remote. The architect ordinarily has considerably less to do with these qualities than the artist, the element of time usually being, for him, of too great importance to permit of the highest finish of which he may be capable; but the fundamental principles upon which they depend, so far as they apply to plane surfaces, with which he is mainly concerned, are the following:

Illuminated surfaces, parallel to the perspective plane, and at different distances, are lightest at the front, and get darker in tone as they recede. When unilluminated, the exact opposite is the rule.

On an illuminated surface seen obliquely, the lightest part is nearest the observer. This rule is also reversed, like the preceding one, for a surface in the shade when viewed obliquely.

When the surface receiving a shadow is of the same nature (material) as that casting it, the shade should be darker than the shadow.

The intensity of a shadow diminishes as the shadow lengthens.

Should the student wish to go thoroughly into the technique of free-hand sketching in black and white, he should obtain Linfoot's *Picture Making with Pen and Ink;* while for the beginner in color work as applied to architectural subjects, F. F. Frederick's *Rendering in Sepia* is admirably adapted. With these, and Delamotte's *Art of Sketching from Nature* he will find himself fully advised on every point that can arise in connection with the free-hand part of his professional work.

# *CHAPTER XV.*

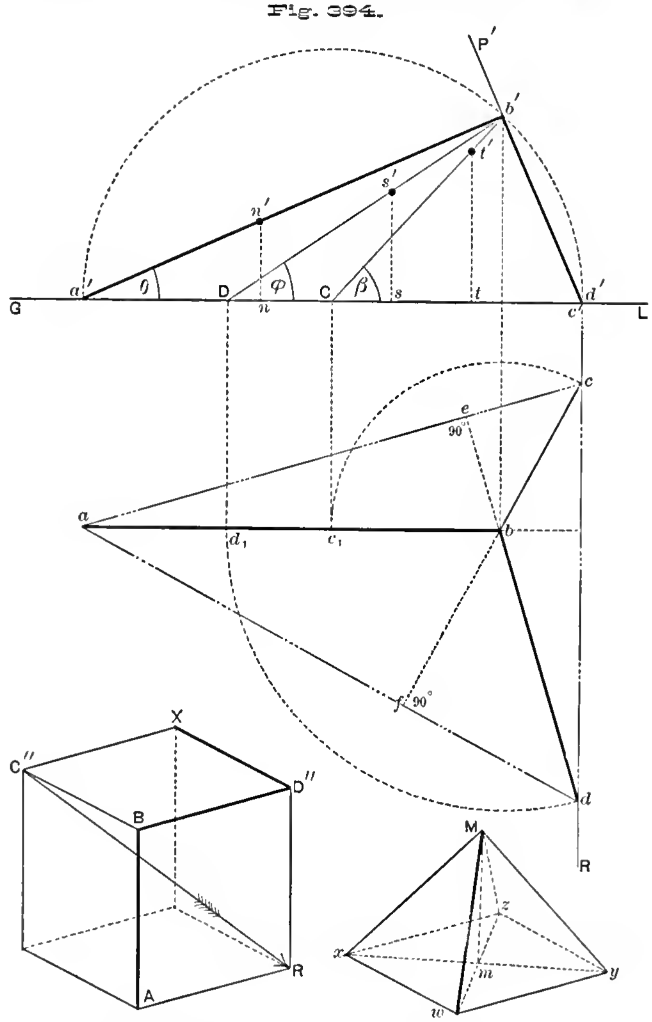
## AXONOMETRIC (INCLUDING ISOMETRIC) PROJECTION.—ONE-PLANE DESCRIPTIVE GEOMETRY.

621. When but one plane of projection is employed there are but two applications of orthographic projection having special names. These are *Axonometric* (known also as *Axometric*) *Projection*, and *One-Plane Descriptive Geometry* or *Horizontal Projection.*

### AXONOMETRIC PROJECTION. — ISOMETRIC PROJECTION.

622. *Axonometric Projection*, including its much-employed special form of *Isometric Projection*, is applicable to the representation of the parts or "details" of machinery, bridges or other constructions in which the main lines are in directions that are mutually perpendicular to each other.

Fig. 394.




An axonometric drawing has a pictorial effect that is obtained with much less work than is involved in the construction of a true perspective, yet which answers almost as well for the conveying of a clear idea of what the object is; while it may also be made to serve the additional purpose of a working drawing, when occasion requires.

623. *Fundamental Problem.—To obtain the orthographic projection of three mutually perpendicular lines or axes, and the scale of real to projected lengths.* Let $a\,b$, $b\,c$ and $b\,d$ (Fig. 394) be the projections of three lines forming a solid right angle at $b$. Let the line $a\,b$ be inclined at some given angle $\theta$ to the plane of projection. Locate a vertical plane parallel to $a\,b$ and project the latter upon it at $a'b'$, at $\theta°$ to the horizontal. Since the plane of the other two axes is perpendicular to $a\,b$, $a'b'$, its traces will be $P'd'R$. (Art. 303).

In order to find either $c$ or $d$ we need to know the inclination of the axis having such point for its extremity. Supposing $\beta$ given for $c\,b$, draw $b'C$ at $\beta°$ to G L; project $C$ to $c_1$ and draw arc $c_1 c$, centre $b$, obtaining $c$.

Join $a$ with $c$; then $a\,c$ is the trace of the plane of the axes $b\,a$ and $b\,c$, and being perpendicular to the third axis we may draw the latter as the line $e\,b\,d$, making 90° with $a\,c$.

Carry $d$ to $d_1$ about $b$; project $d_1$ to $D$ and join the latter with $b'$. Then $Db'$ is the *true length*, and $b'DL$ (or $\phi$) the *inclination*, of the third axis, $bd$.

Lay off $a'n'$, $Ds'$ and $Ct'$, each one inch. Their projected lengths on the horizontal are respectively $a'n$, $Ds$ and $Ct$. The latter are then the lengths, representative of inches, for all lines parallel to $ab$, $bc$ and $bd$ respectively.

624. *To make an axonometric projection of a one-inch cube*, to the scale just obtained.

Although not absolutely necessary *it is customary to take one axis vertical.*

Taking the $ab$-axis vertical, the cube in Fig. 394 fulfills the conditions. For $BA$ equals $a'n$; $BD''$ equals $Ds$, and $BC''$ equals $Ct$, while the angles at $B$ equal those at $b$.

The light being taken in the usual direction, i. e., parallel to the body-diagonal of the cube ($C''R$), the shade lines indicated are those which separate illumined from unillumined surfaces, and are those which could, therefore, cast shadows.

625. *The axonometric projection of a vertical pyramid*, of three-fourths-inch altitude and inch-square base, to the same scale as the cube. The pyramid in Fig. 394 meets the requirements, $xwyz$ having been made equal to $C''BD''X$; while the altitude $mM$, rising from the intersection of the diagonals of the base, equals three-fourths $a'n$, the inch-representative for the vertical axis.

626. *To draw curves in axonometric projection* obtain first the projections of their inscribed or circumscribed polygons, or of a sufficient number of secant lines; then sketch the curve through the points on these new lines which correspond to the points common to the curves and lines in the original figure. This will be illustrated fully in treating isometric projection.

627. *Isometric Projection.—Isometric Drawing.* When three mutually perpendicular axes are *equally inclined* to the plane of projection they will obviously make equal angles (120°) with each other in projection. This relation led to the name "isometric," implying equal measure, and also obviates the necessity for making a separate scale for each axis.

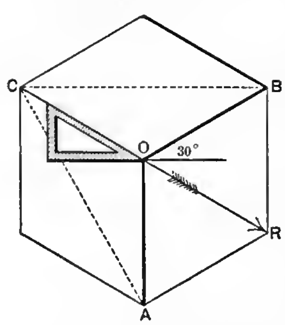
The advantages of this method seem to have been first brought out by Prof. Farish of England, who presented a paper upon it in 1820 before the Cambridge Philosophical Society of England.

628. In practice *the isometric scale is never used*, but, as all lines parallel to the axes are equally foreshortened, it is customary to lay off their given lengths directly upon the axes or their parallels, the result showing relative position and proportion of parts just as correctly as a true projection, but being then called an *isometric drawing*, to distinguish it from the other. It would, obviously, be the *projection* of a considerably larger object than that from which the dimensions were taken.

Lines parallel to the axes are called *isometric lines*.

Any plane parallel to, or containing two isometric axes, is called an *isometric plane*.

Fig. 395.




629. *To make an isometric drawing of a cube of three-quarter-inch edges.* Starting with the usual *isometric centre*, $O$, (Fig. 395) draw one axis vertical, and on it lay off $OA$ equal to three-fourths of an inch. $OC$ and $OB$ are then drawn with the 30°-triangle as shown, made equal in length to $OA$, and the figure completed by parallels to the lines already drawn.

One body-diagonal of the cube is perpendicular to the paper at $O$.

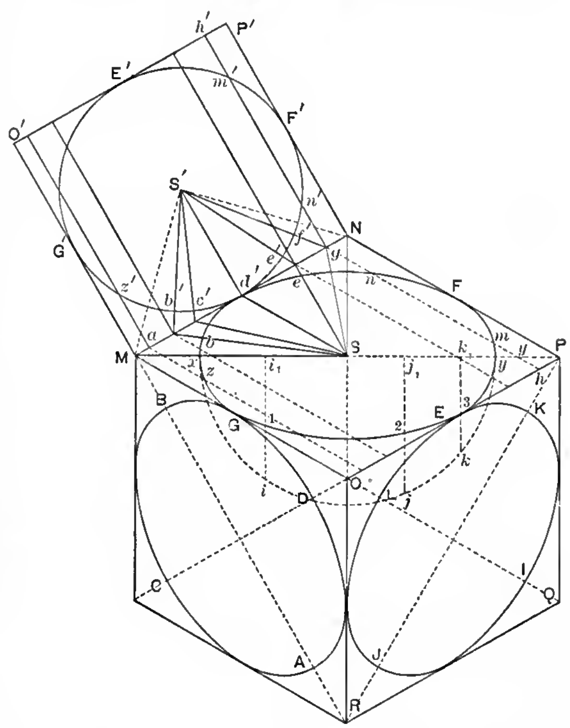
630. *To draw circles and other curves isometrically*, employ auxiliary tangents and secants, obtain their isometric representations, and sketch the curves through the proper points.

In Fig. 396 we have an isometric cube, and at $MO'P'N$ the square, which—by rotation on $MN$ and by an elongation of $MP'$—becomes transformed into $MOPN$. The circle of centre $S'$ then

becomes the ellipse of centre $S$, whose points are obtained by means of the four tangencies $d'$, $F$, $E$ and $G$, and by making $g\,n$ equal to $g\,n'$, $h\,m$ equal to $h'm'$, etc.

631. *The isometric circle may be divided* into parts corresponding to certain arcs on the original, either (1) by drawing radii from $S'$ to $MN$, as those through $b'$, $c'$, $d'$, (which may be equidistant or not, at pleasure) and getting their isometric representatives, which will intercept arcs, as $b\,d'$, $d'e$, which are the isometric views of $b'd'$, $d'e'$; or (2) by drawing a semicircle $x\,i\,y$ on the major axis as a diameter, letting fall perpendiculars to $x\,y$ from various points, and noting the arcs as 1–2, 2–3, that are included between them and which correspond to the arcs $i\,j$, $j\,k$, originally assumed.

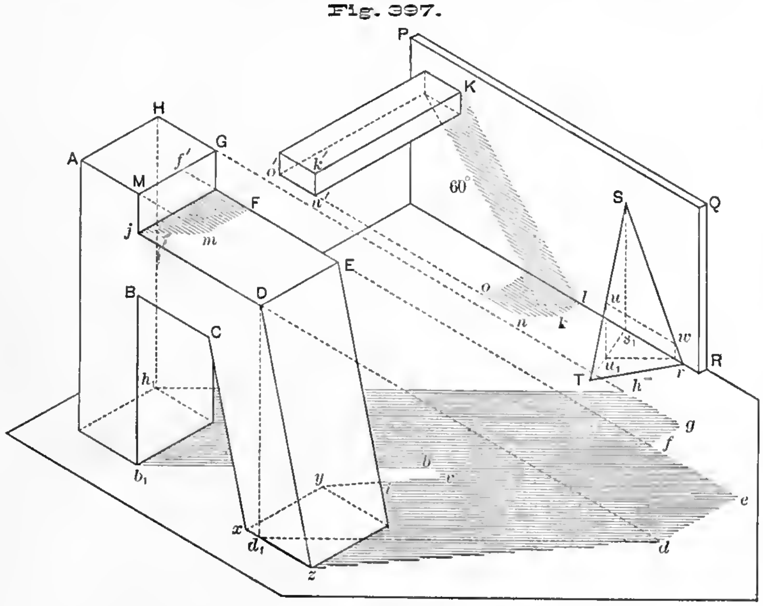
Fig. 396.



632. *Shade lines on isometric drawings.* While not universally adhered to, the conventional direction for the rays, in isometric shadow construction, is that of the body-diagonal $C\,R$ of the cube (Fig. 395). This makes in projection an angle of 30° with the horizontal. Its projection on an isometrically-horizontal plane—as that of the top—is a horizontal line $C\,B$; while its projection $C\,A$, on the isometric representation of a vertical plane, is inclined 60° to the horizontal.

633. To illustrate the principles just stated Fig. 397 is given, in which all the lines are isometric, with the exception of $D\,z$ and its parallels, and $S\,T$. The drawing of non-isometric lines will be treated in the next article, but assuming the objects as *given* whose shadows we are about to construct, we may start with any line, as $D\,z$.

Fig. 397.



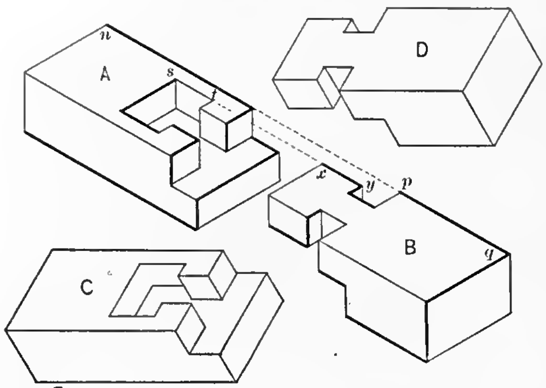
The ray $D\,d$ is at 30°. Its projection $d_1 d$ is a horizontal through the plan of $D$. The ray and its projection meet at $d$. As the shadow begins where the line meets the plane, we have $z\,d$ for the shadow of $D\,z$. This gives the direction for the shadow of any line parallel to $D\,z$, hence for $y\,v$, which, however, soon runs into the shadow of $B\,C$. As $b$ is the intersection of the ray $B\,b$ with its projection $b_1 b$, it is the shadow of $B$, and $b_1 b$ that of $b_1 B$. Then $b\,v$ is parallel to $B\,C$, the line casting the shadow being parallel to the plane receiving it.

In accordance with the principle last stated, $d\,e$ is equal and parallel to $D\,E$, and $e\,f$ to $E\,F$. At $f$ the shadow turns to $g$, as the ray $f\,F$, run back, cuts $M\,G$ at $f'$, and $f'G$ casts the $f\,g$-shadow.

Then $gh$ equals $GH$, and $hh_1$ is the shadow of $Hh$. The projection $jm$ catches the ray $Mm$ at $m$. Then $mf$, equal to $Mf'$, completes the construction.

The timber, projecting from the vertical plane $PQR$, illustrates the 60°-angle earlier mentioned. $Kk'$ being perpendicular to the vertical plane, its shadow $Kl$ is at 60° to the horizontal, and $Klk$ is the plane of rays containing said edge. Its horizontal trace catches the ray from $k'$ at $k$. Then $nk$, the shadow of $n'k'$, is horizontal, being the trace of a vertical plane of rays on an isometrically-horizontal plane. The construction of the remainder is self-evident.

Fig. 398.



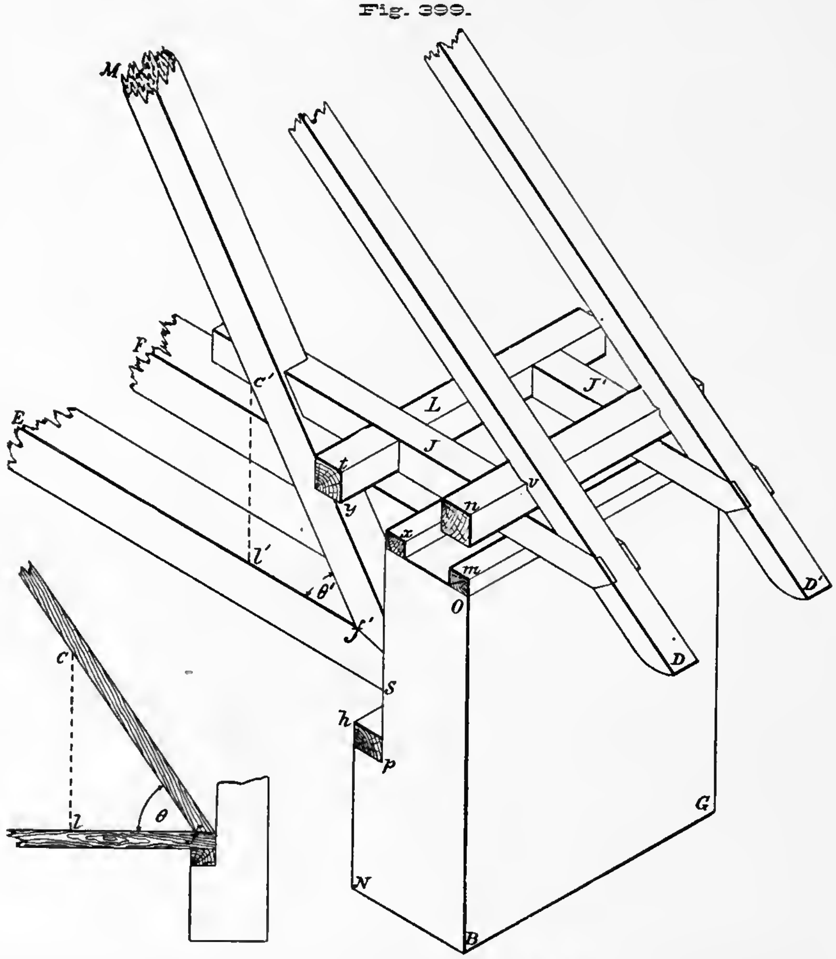

Letting $ST$ represent a small rod, oblique to isometric planes, assume any point on it, as $u$; find its plan, $u_1$; take the ray through $u$ and find its trace $w$. Then $Sw$ is the direction of the shadow on the vertical plane, and at $r$ it runs off the vertical and joins with $T$.

634. *Timber framings, drawn isometrically*, are illustrated by Figures 398 and 399. In Fig. 398 the pieces marked $A$ and $B$ show one form of mortise and tenon joint, and are drawn with the lines in the customary directions of isometric axes. The same pieces are represented again at $C$ and $D$, all the lines having been turned through an angle of 30°, so that while maintaining the same relative direction to each other and being still truly isometric, they lie differently in relation to the edges of the paper—a matter of little importance when dealing with comparatively small figures, but affecting the appearance of a large drawing very materially.

635. *Non-isometric lines.—Angles in isometric planes.* In Fig. 399 a portion of a cathedral roof truss is drawn isometrically.

Three pieces are shown that are not parallel to isometric lines. To represent them correctly we need to know the real angles made by them with horizontal or vertical pieces, and use isometric coördinates or "offsets" in laying them out on the drawing.

Fig. 399.




In the lower figure we see at $\theta$ the actual angle of the inclined piece $Mf'$ to the horizontal. Offsets, $fl$ and $lC$, to any point $C$ of the inclined piece, are laid off in isometric directions at $f'l'$

and $l'C'$, when $C'f'l'$ (or $\theta'$) is the isometric view of $\theta$. A similar construction, not shown, gave the directions of pieces $D$ and $D'$.

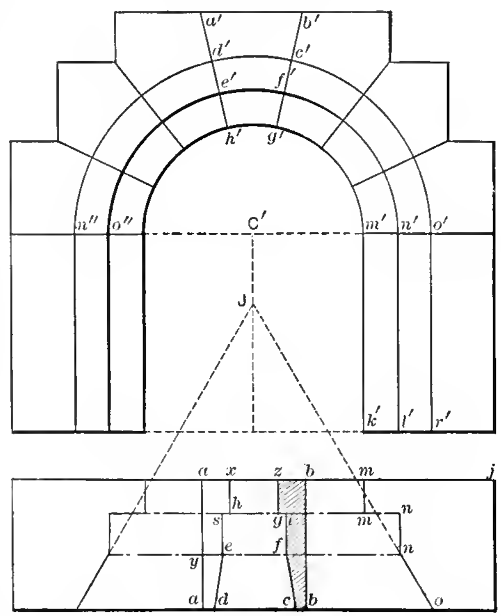
Much depends on the choice of the *isometric centre.* Had $N$ been selected instead of $B$, the top surfaces of the inclined pieces would have been nearly or quite projected in straight lines, rendering the drawing far less intelligible.

The student will notice that the shade lines on Fig. 399 are located *for effect,* and in violation of the usual rule, it having been found that the best appearance results from assuming the light in such direction as to make the most shade lines fall centrally on the timbers.

636. *Non-isometric lines.—Angles not in isometric planes.* To draw lines not lying in isometric planes requires the use of three isometric offsets. As one of the most frequent applications of isometric drawing is in problems in stone cutting, we may take one such to advantage in illustrating constructions of this kind.

Fig. 400 shows an arched passage-way, in plan and elevation. The surface $n\,o$, $r'l'n'o'$ is vertical as far as $n'o'$, and conical (with vertex $J$, $C'$) from there to $n''o''$. The vertical surface on $n\,n$ is tangent at $n'$ to the cylinder $n'f'e'o''$. Similarly, $m\,m$ is vertical to $m'$, and there changes into the cylinder $m'g'h'$.

Fig. 400.

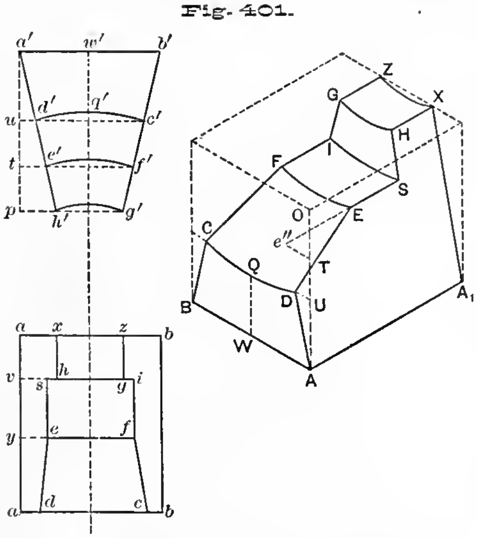



The radial bed $b'g'$ is indicated on the plan (though not in full size) by parallel lines at $b\,c\,f\,i\,g\,z\,b$. The bed $a'h'$ is of the same form as $b'g'$, being symmetrical with it.

In Fig. 401 we have an enlarged drawing of the keystone with the plan inverted, so that all the faces of the stone may be correctly represented as seen. The isometric drawing is made to correspond, that is, it represents the stone after a 180°-rotation about an axis perpendicular to the paper.

The isometric block in which the keystone can be inscribed is shown in dotted lines, its dimensions, derived from the projections, being *length,* $A\,A_1 = a\,a$; *breadth,* $A\,B = a'b'$; *height,* $A\,O = a'p$.

Fig. 401.




The top surface $a'b'$ becoming the lower in the isometric, reverses the direction of the lines. Thus, $a'$ is seen at $A$, and $b'$ at $B$. To get $D$ make $A\,U = a'u$, then $U\,D = u\,d'$. Make $C$ symmetrical with $D$ and join with $B$, and also $D$ with $A$. $W\,Q$ equals $w'q'$, for the ordinate of the middle point of the arc.

$D\,E$ is not an isometric plane, hence to reach $E$ from $A$ we make $A\,T = a't$; $T\,e'' = t\,e'$, and $e''E = a\,y$ (the distance of $e$ from the plane $a\,b$).

The remainder of the construction is but a duplication of one or other of the above processes.

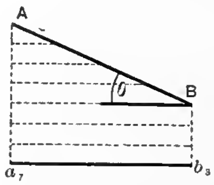
The principle that lines that are parallel on the object will also be parallel on the drawing may be frequently availed of in the interest of rapid construction or for a check as to accuracy.

### HORIZONTAL PROJECTION OR ONE-PLANE DESCRIPTIVE GEOMETRY.

637. *One-Plane Descriptive Geometry* or *Horizontal Projection* is a method of using orthographic projections with but one plane, the fundamental principle being that the space-position of a point is known if we have its projection on a plane and also know its distance from the plane.

Thus, in Fig. 402, $a$ with the subscript 7 shows that there is a point $A$, vertically above $a$ and at seven units distance from it. The significance of $b_3$ is then evident, and to show the line in its true length and inclination we have merely to erect perpendiculars $aA$ and $Bb$, of *seven* and *three* units respectively, join their extremities, and see the line $AB$ in true length and inclination.

Fig. 402.

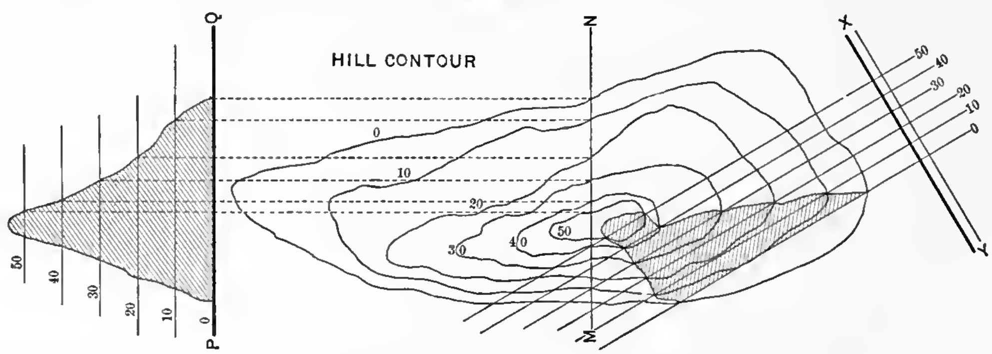



In this system the horizontal plane alone is used; One-plane Descriptive is therefore applied only to constructions in which the lines are mainly or entirely horizontal, as in the mapping of small topographical or hydrographical surveys, in which the curvature of the earth is neglected; also in drawing fortifications, canals, etc.

The plane of projection, usually called the *datum* or *reference* plane, is taken, ordinarily, below all the points that are to be projected, although when mapping the bed of a stream or other body of water it is generally taken at the water line, in which case the numbers, called *indices* or *references*, show depths.

638. A *horizontal* line evidently needs but one index. This is illustrated in mapping *contour lines*, which represent sections of the earth's surface by a series of equidistant horizontal planes.

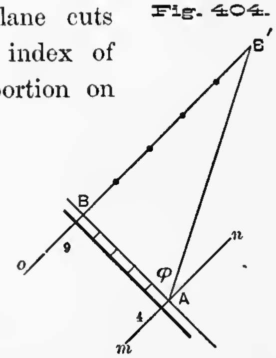
Fig. 403.




In Fig. 403 the curves indicate such a series of sections made by planes one yard, metre or other unit apart, the larger curve being assumed to lie in the reference or datum plane, and therefore having the index *zero*.

The profile of a section made by any vertical plane $MN$ would be found by laying off—to any assumed scale for vertical distances—ordinates from the points where the plane cuts the contours, giving each ordinate the same number of units as are in the index of the curve from which it starts. Such a section is shown in the shaded portion on the left, on a ground line $PQ$, which represents $MN$ transferred.

Fig. 404.




639. The steepness of a plane or surface is called its *slope* or *declivity*. A *line of slope* is the steepest that can be drawn on the surface. *A scale of slope* is obtained by graduating the plan of a line of slope so that each unit on the scale is the projection of the unit's length on the original line. Thus, in Fig. 404, if $mn$ and $oB$ are horizontal lines in a plane, one having the index 4 and the other 9, the point $B$ is evidently five units above $A$, and the five equal divisions between it and $A$ are the projections of those units.

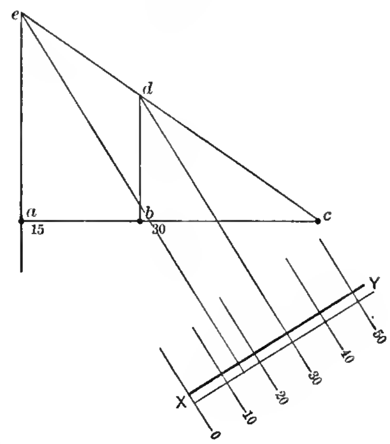
The scale of slope is often used as a ground line upon which to get an edge view of the plane. Thus, if $BB'$ is at $90°$ to $BA$, and its length five units, then $B'A$ is the plane, and $\phi$ is its inclination.

The scale of slope is always made with a double line, the heavier of the two being on the left, ascending the plane.

As no exhaustive treatment of this topic is proposed here, or, in fact, necessary, in view of the simplicity of most of the practical applications and the self-evident character of the solutions, only two or three typical problems are presented.

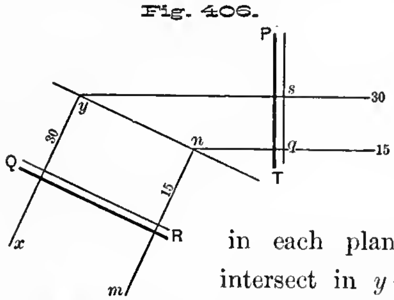
640. *To find the intersection of a line and plane.* Let $a_{15} b_{30}$ be the line, and $XY$ the plane. Draw horizontal lines in the plane at the levels of the indexed points. These, through 15 and 30 on $XY$, meet horizontal lines through $a$ and $b$ at $e$ and $d$; $ed$ is then the line of intersection of $XY$ and a plane containing $ab$; hence $c$ is the intersection of the latter with $XY$.

Fig. 405.




The same point $c$ would have resulted if the lines $ae$ and $bd$ had been drawn in any other direction while still remaining parallel.

Fig. 406.




641. *To obtain the line of intersection of two planes,* draw two horizontals in each, at the same level, and join their points of intersection.

In Fig. 406 we have $mn$ and $qn$ as horizontals at level 15, one in each plane. Similarly, $xy$ and $ys$ are horizontals at level 30. The planes intersect in $yn$.

Were the scales of slope parallel, the planes would intersect in a horizontal line, one point of which could be found by introducing a third plane, oblique to the given planes, and getting its intersection with each, then noting where these lines of intersection met.

642. *To find the section of a hill by a plane of given slope.* Draw, as in the problem of Art. 640, horizontal lines in the plane, and find their intersections with contours at the same level. Thus, in Fig. 403, the plane $XY$ cuts the hill in the shaded section nearest it, whose outlines pass through the points of intersection of horizontals 10, 20, 30 of the plane, with the like-numbered contour lines.

# CHAPTER XVI.

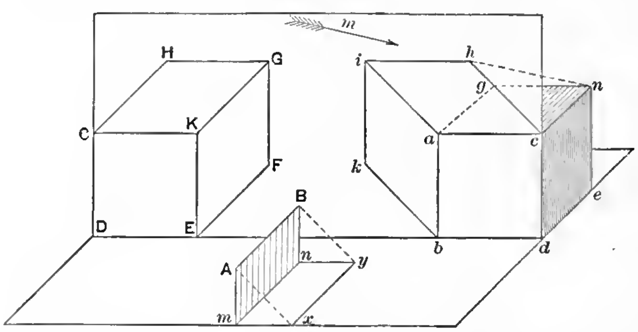
## OBLIQUE OR CLINOGRAPHIC PROJECTION.—CAVALIER PERSPECTIVE.—CABINET PROJECTION. MILITARY PERSPECTIVE.

643. If a figure be projected upon a plane by a system of parallel lines that are oblique to the plane, the resulting figure is called an *oblique* or *clinographic projection*, the latter term being more frequently employed in the applications of this method to crystallography. Shadows of objects in the sunlight are, practically, oblique projections.

In Fig. 407, $ABnm$ is a rectangle and $mxyn$ its oblique projection, the parallel projectors $Ax$ and $By$ being inclined to the plane of projection.

644. *When the projectors make* 45° *with the plane* this system is known either as *Cavalier Perspective, Cabinet Projection* or *Military Perspective*, the plane of projection being vertical in the case of the first two, and horizontal in the last.

Fig. 407.




645. *Cavalier Perspective.—Cabinet Projection.—Military Perspective.* As just stated, the projectors being inclined at 45° for the system known by the three names above, we note that in this case a line perpendicular to the plane of projection, as $Am$ or $Bn$ (Fig. 407), will have a projection equal to itself. It is, therefore, unnecessary to draw the rays for lines so situated, as the known original lengths can be directly laid out on lines drawn in the assumed direction of projections.

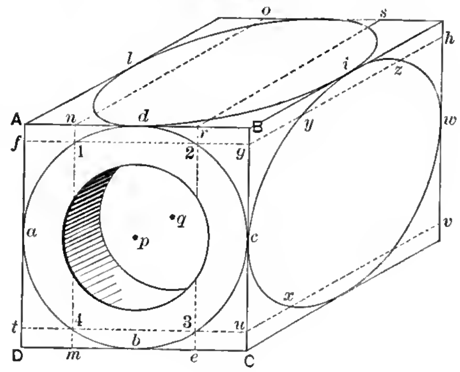
Let $abcd.n$ be a cube with one face coinciding with the vertical plane. If the arrow $m$ indicates one direction of rays making 45° with V, then the ray $hn$, parallel to $m$, will give $h$ as the projection of $n$, and from what has preceded we should have $ch$ equal to $cn$, and analogously for the remaining edges, giving $abcd.i$ for the cavalier perspective of the cube.

Similarly, $EKH$ is a correct projection of the same cube for another direction of projectors, and we may evidently draw the oblique edges in any other direction and still have a cavalier perspective, by making the projected line equal to the original, when the latter is perpendicular to the plane of projection.

646. *Oblique projection of circles.* Were a circle inscribed in the back face of the cube $DKG$ (Fig. 407) the projectors through the points of the circle would give an oblique *cylinder of rays*, whose intersection with the vertical plane $DX$ would be a circle, since parallel planes cut a cylinder in similar sections. We see, therefore, that the oblique projection of a circle is itself circular when the plane of projection is parallel to that of the circle. In any other case the oblique projection of a circle may be found like an isometric projection (see Art. 631), viz., by drawing chords of the circle, and tangents, then representing such auxiliary lines in oblique view and sketching the curve (now an ellipse) through the proper points. Fig. 408 illustrates this in full.

647. *Oblique projection* is even better adapted than isometric to the representation of timber framings, machine and bridge details, and other objects in which straight lines—usually in mutually perpendicular directions—predominate, since all angles, curves, etc., lying in planes parallel to the paper, appear of the same *form* in projection, while the relations of lines perpendicular to the paper are preserved by a simple ratio, ordinarily one of equality.

Fig. 408.




648. When the rays make with the plane of projection an angle greater than 45°, oblique projections give effects more closely analogous to a true perspective, since the foreshortening is a closer approximation to that ordinarily existing from a finite point of view. This is illustrated by Fig. 409, in which an object $ABDE$, known to be 1″ thick, has its depth represented as only $\frac{1}{4}$″ in the second view, instead of full size, as in a cavalier perspective, the front faces being the same size in each. Provided that the scale of reduction were known, $abcdkf$ would answer as well for a working drawing as a 45°-projection.

649. By way of contrast with an isometric view the timber framing represented by Fig. 398 is

Fig. 409.

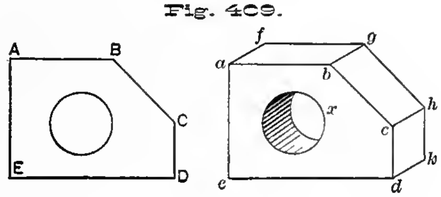



Fig. 410.

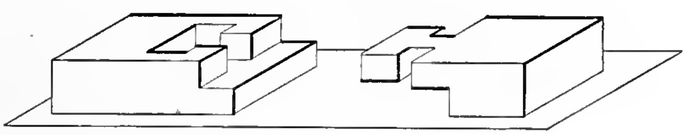
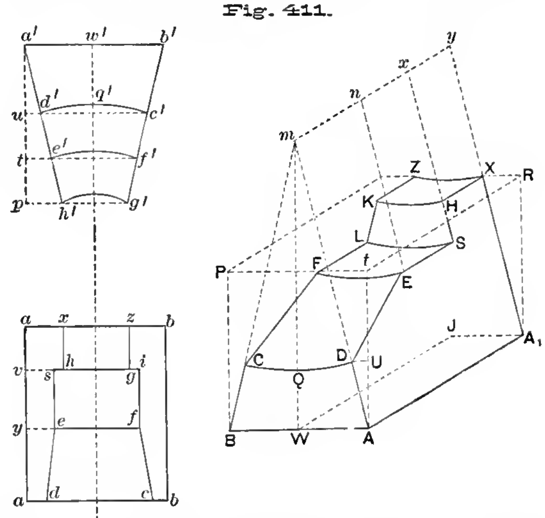


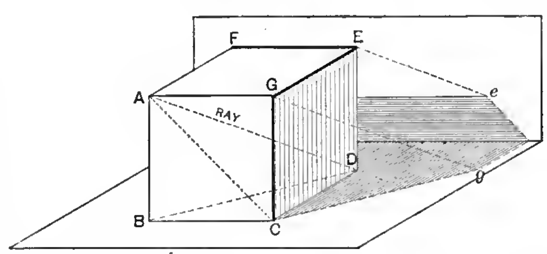
Fig. 411.




drawn in cavalier perspective in Fig. 410. Reference may advantageously be made, at this point, to Figs. 44, 45 and 46, which are oblique views of one form.

The keystone of the arch in Fig. 400, whose isometric view is shown in Fig. 401, appears in oblique projection in Fig. 411; the direction of lines not parallel to the axes of the circumscribing prism being found by "offsets" that must be taken in axial directions.

Fig. 412.

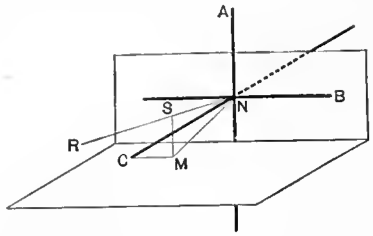



650. *Shadows, in oblique projection.* As in other projections, the conventional direction for the light is that of the body-diagonal of the oblique cube. The edges to draw in shade lines are obvious on inspection. (Fig. 412).

651. An interesting application of oblique projection, earlier mentioned, is in the drawing of crystals. Fig. 414 illustrates this, in the representation of a form common in fluorite and called the tetrahexahedron, bounded by twenty-four planes, each of which fulfills the condition expressed in the formula

$\infty : n : 1$; that is, each face is parallel to one axis, cuts another at a unit's distance, and the third at some multiple of the unit.

Fig. 413.

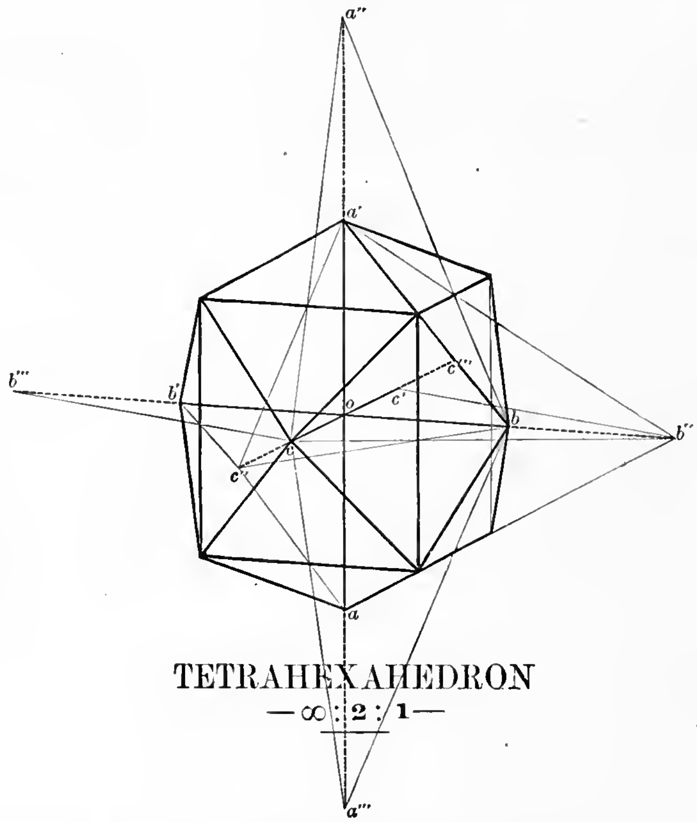



The three axes in this system are equal, and mutually perpendicular; but their projected lengths are $a\,a'$, $b\,b'$, $c\,c'$.

The direction of projectors which was assumed to give the lengths shown, was that of $RN$ in Fig. 413, derived by turning the perpendicular $CN$ through a horizontal angle $CNM = 18°\,26'$, and then elevating it through a vertical angle $MNS = 9°\,28'$; values that are given by Dana as well adapted to the exhibition of the forms occurring in this system.

The axes once established, if we wish to construct on them the form $\infty : 2 : 1$, we lay off on

Fig. 414.




each (extended) one-half its own (projected) length; thus $c\,c''$ and $c'c'''$ each equal $o\,c'$; $b\,b''$ equals $o\,b$, etc. Then draw in light lines the traces of the various faces on the planes of the axes. Thus, $a'b''$ and $a''b$ each represent the trace of a plane cutting the $c$-axis at infinity, and the other axes at either one or two units distance; the former intercepting the *two* units on the $b$-axis and the *one* on the $a$-axis, while for $a''b$ it is exactly the reverse. Through the intersection of $a'b''$ and $a''b'$ a line is drawn parallel to the $c$-axis, indefinite in length at first, but determinate later by the intersection with it of other edges similarly found.

The student may develop in the same manner the forms $\infty : 3 : 1$; $\infty : 2 : 3$; $\infty : 3 : 4$; $\infty : 4 : 5$.

# CHAPTER XVII.

END-POST BRIDGE DETAIL, UPPER CHORD.—SPUR GEAR, APPROXIMATE INVOLUTE OUTLINES. HELICAL SPRINGS.—STANDARD BOLTS, SCREWS AND NUTS.—TABLE OF PROPORTIONS.

Supplementing the working drawings found in the earlier pages, a few others of frequent occurrence are given in this chapter, in concluding the main portion of the book. In the Appendix will be found one or two more, not needing descriptive text,—an 100-pound rail section and an Allen-Richardson valve, the latter of the proportions employed on one of the locomotives drawing the celebrated "Exposition Flyer" in 1892; also a table of the proportions of washers.

It need hardly be said that the illustrations are not to be copied by transfer with dividers, but that to get all the benefit intended from their presentation they should have their proportions reduced to scale by the student, in which case the work becomes in greater degree constructive, and in closer analogy to that of the shop draughting-room, which is so frequently from free-hand, dimensioned sketches.

652. *Detail of a Bridge.—Upper-Chord Post-Connection.*—A bridge or roof truss is an assemblage of pieces of iron or wood, so connected that the entire combination acts like a single beam. Figs. 415 and 416 are what are called "skeleton diagrams" of bridge trusses, each piece or "member" of the truss being represented by a single line. *A B C D* and *A′ B′ C′ D′* are the trusses proper, the

Fig. 415.

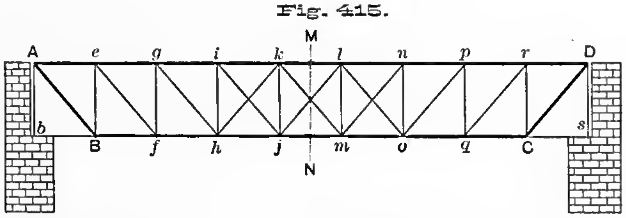
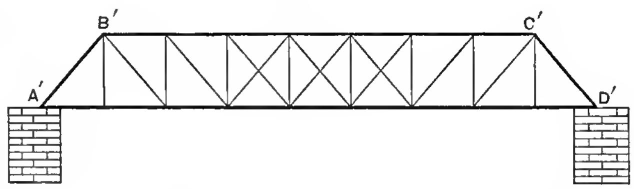



Fig. 416.




former being for an overhead track and the latter for a roadway running through the bridge. In each case the upper part—called the *upper chord*—(*A D*, *B′ C′*) sustains *compression*, and is made of "built beams," formed by riveting together various plates and lengths of structural iron in such manner as to form one practically continuous column.

The *lower chords* (*B C*, *A′ D′*) sustain tension, and are made of bars of high tensile strength.

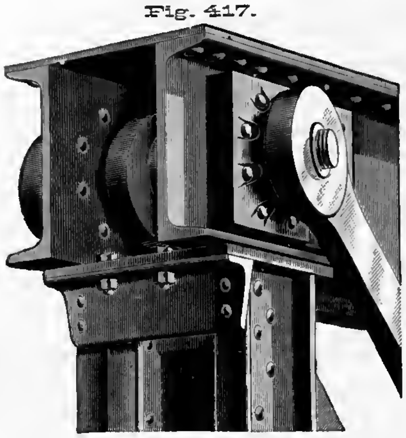
The members that connect the chords are called either *ties* or *struts* according as the strain in them is tensile or compressive. Collectively they form the *web* of the truss.

In the form of truss illustrated—which is only one of many which have commended themselves to the profession—the vertical pieces or "posts," *B e*, *f g*, etc., sustain compression, and are therefore "built" columns. They divide the trapezoid into parts called *panels*, which has given the name *panel system* to this largely-employed arrangement of bridge members.

All the diagonal members in both figures, excepting *A′ B′* and *C′ D′*, are tension bars or rods. *B b* and *C s* are struts whose sole office is to keep the posts *A b* and *D s* vertical; said posts then conveying to the masonry whatever weights are transmitted through the truss to *A* and *D* respectively.

Fig. 417 is a perspective view of the connection at $A$ between the post $A\,b$, the upper chord $A\,D$, and the four diagonal bars that are projected in $A\,B$. The working drawings required for such connection are shown in the three views on the opposite page. Three analogous views would be required for the connection at the foot of the same post.

Fig. 417.



When—as in the case from which our example is taken—there are *two* railroad tracks overhead, the members of the middle truss will usually have different proportions from those in the outside trusses, and a separate set of three views has therefore to be made for each of its post connections, so that the smallest number of shop drawings for one such bridge—after making all allowance for the symmetry of the structure with reference to the central plane $M\,N$—would consist of twenty such groups of three as are illustrated by Fig. 419.

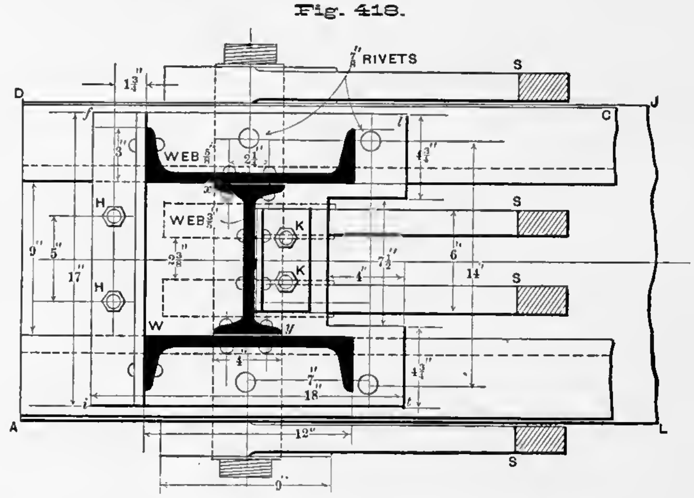
The upper projections (Fig. 419) are obviously a front and a side elevation. The lower figure may preferably be regarded as a plan of the object inverted, since that conception is somewhat more natural than that of the post in its normal position, while the draughtsman lies on his back and gazes up at it from beneath.

Fig. 418 shows the inverted plan on a somewhat smaller scale, and, although presented mainly to illustrate the contrast between views with shade lines and without, contains one or two serviceable dimensions that are omitted on the other plate.

653. *General description.* Referring to the wood cut as well as the orthographic projections, we find the upper chord to be composed of a long cover-plate, $18'' \times \frac{7}{16}''$, riveted to the top angles of two vertical channel bars set back to back; each channel being 15″ high and weighing 200 pounds per yard. The cross section of the upper chord is shown in solid black, with just enough space intended between plate and channels to show that they are not all in one piece.

Perpendicular to the vertical faces of the channels and through holes cut therein runs a cylinder called a "pin," 4″ in diameter and $21\frac{3}{4}''$ "between shoulders" (as marked on the plan), that is, between the planes where the diameter is reduced and a thread turned, on which connection can be made with the corresponding post in the next truss.

Fig. 418.




Four diagonal bars are sustained by the pin, the latter passing through holes, called "eyes," in the bars. Two of the bar heads are between the channels.

Two plates are inserted between each of the outside bars and the nearest channel not only to prevent the bar from touching the angle, as at $h$, but also to relieve the metal nearest the pin from some of the strain. The longer plate $m\,n\,F\,E$ is next the channel. The other, $n\,m\,o\,p$, has a kind of hub cast on it which rounds up to the bar head, as shown in the side elevation.

The vertical post is made up of an I-beam and two channels, as shown by the black sections on the plan.

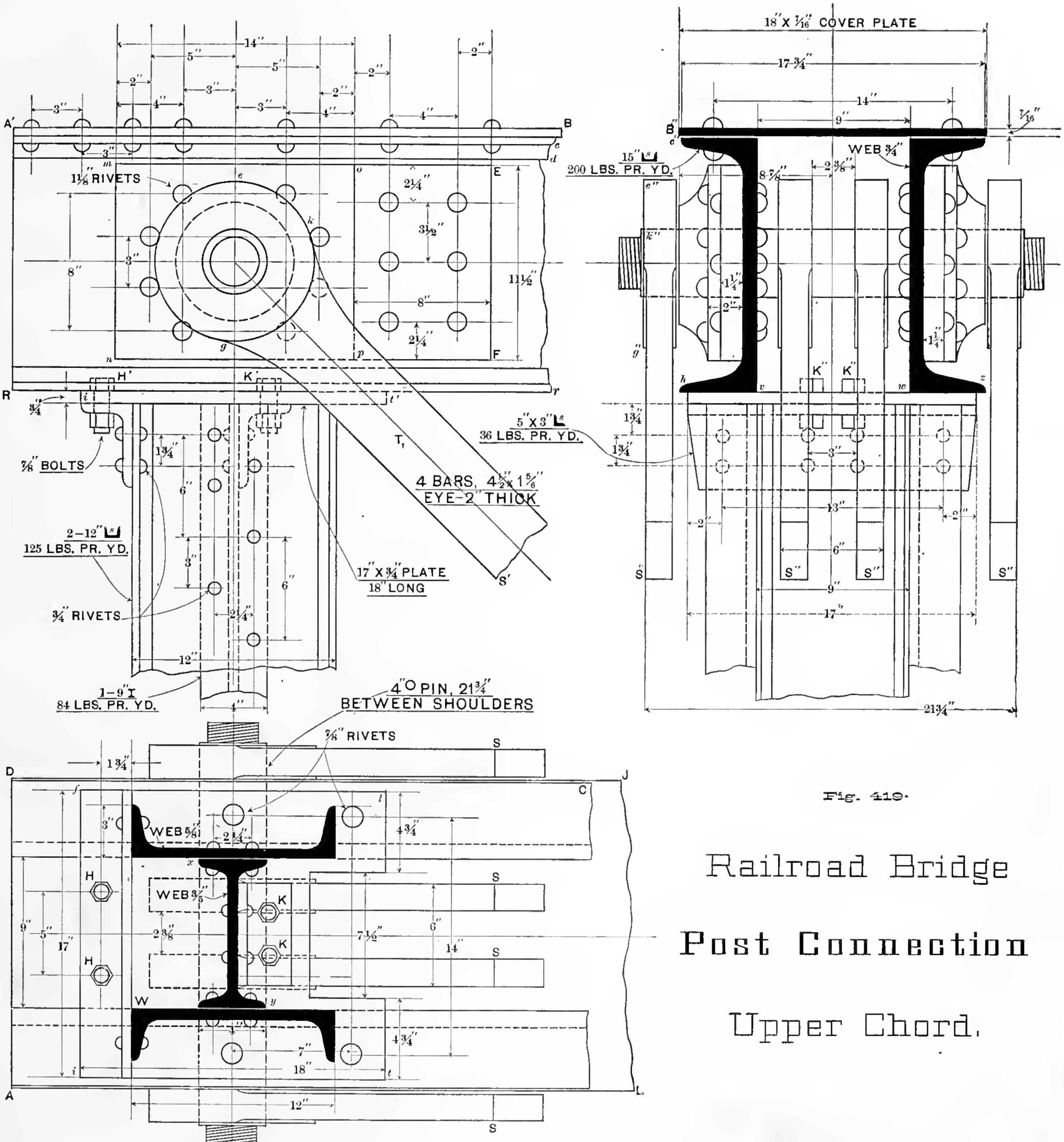



Fig. 419.

Railroad Bridge

Post Connection

Upper Chord.

When drawing the above the student should make *horizontal* dividing lines in all fractions. A brush tint of Prussian blue for all the metal parts will enhance the appearance of the drawing very materially, but the "previous lining" should be, obviously, in best waterproof ink. Centre, dimension, and extension lines should be in continuous red lines, unless for blue printing, in which case all lines will be black.

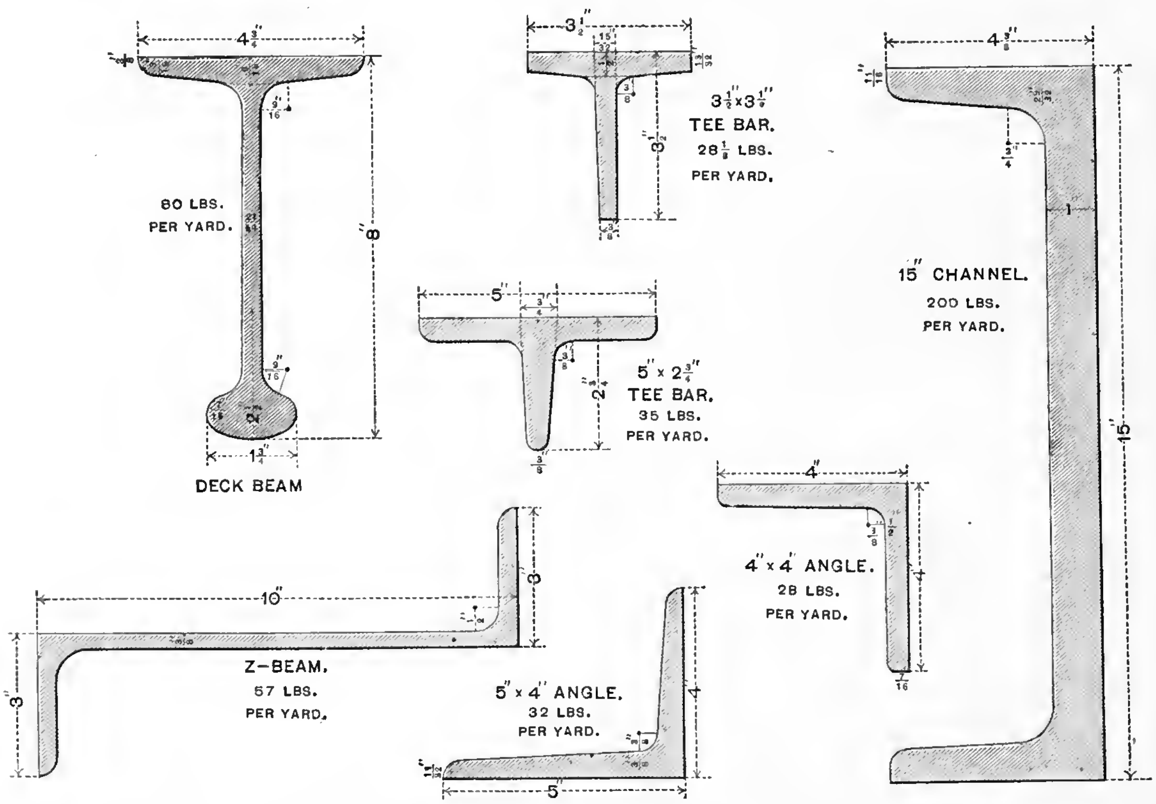
Between the upper chord and the top of the post is a three-quarter-inch plate, seen best on the plans at *i f l t*. It is nicked out 4″, near the nuts *K*, so as to clear the two middle bars *S* which come between the channels.

A 5″ × 3″ angle-iron runs from outside to outside of channels, and is held by rivets and by the bolts marked *H*. A shorter piece of the same kind is fastened by bolts *K* to the plate, and, by rivets to the web of the I-beam.

654. *Hints as to drawing the bridge post connection.* Draw the main centre lines first; then the plan and side elevations simultaneously, as the horizontal centre line of the plan represents the same vertical reference-plane as the vertical centre line of the side elevation, and one spacing of the dividers may be made to do double work.

The solid sections should be drawn first of all; then the pins, bars, and cap plates of the post in the order named. The parts already drawn should next be represented on the front elevation by

Fig. 420.




projecting up from the plan and across from the side view. The filler plates, *m p* and *m F*, with their rivets, come next on the front elevation, from which they project to the side.

Next in order draw the angle irons on the front elevation, with their bolts, *H′* and *K′*, and project them both across and down. Finally put in all remaining rivets, and dimension the views.

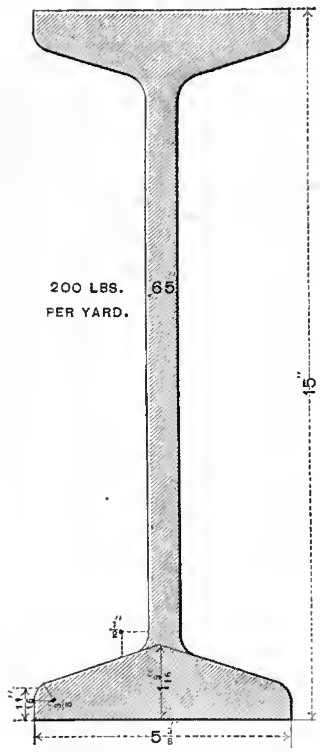
The angle whose bolts are marked *H* terminates exactly on the edges of the channels, as shown in the wood-cut, rather than as indicated in the side elevation.

655. *Structural Iron.* In Figs. 420 and 421 the forms of iron more generally used in bridge and house construction are shown in cross section, and may advantageously be drawn on an enlarged scale.

Treated as described in Art. 75 they may be worked up with brush or pen like Fig. 136.

Fig. 421.

15" I-BEAM.




656. *Toothed Gearing.* When two shafts are to be rotated and a constant velocity ratio maintained between them, it is customary to fix upon them toothed wheels whose teeth are so proportioned that by their sliding action upon each other they produce the motion desired. It is not within the intended scope of this work to go at length into the theory of gearing, for which the student is referred to such specialized treatises as those of Grant, Robinson, MacCord, Weisbach and Willis; but the draughtsman will find it to his advantage to be familiar with the following rapid method of drawing the outlines of the teeth of a spur wheel, in which a remarkably close approximation is made by circular arcs to the theoretical involute outlines now so much employed.

657. $CN$ (in Fig. 422) is the radius of the *pitch circle*, that is, the circle which passes through the middle of the working part of the tooth.

The working outlines outside the pitch circle are called *faces* ($fg$, $hi$), while *within* they are designated as *flanks.* The flanks are rounded off into the *root circle* by small arcs called *fillets.*

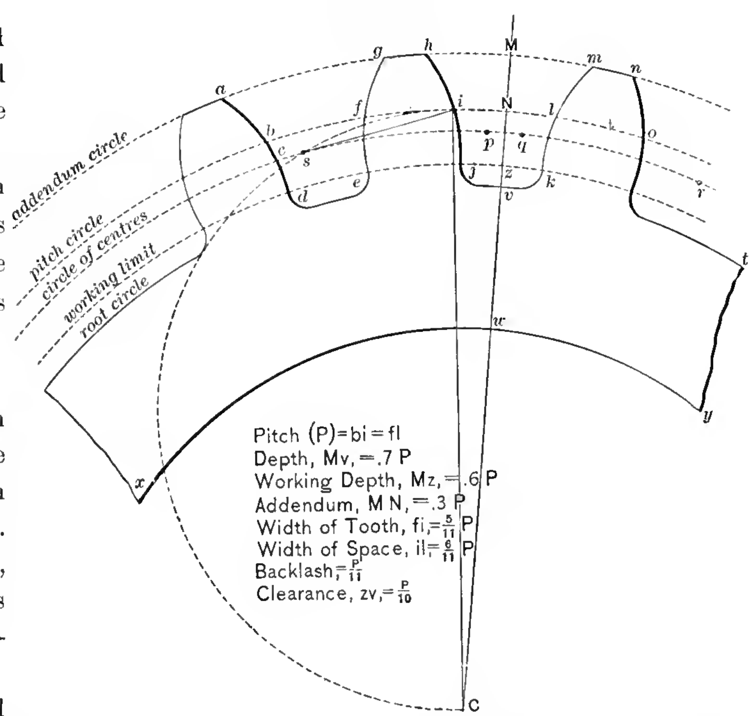
The limits of the teeth on the addendum circle, as $a$, $g$, $h$, $m$, are called their *points.*

On the pitch circle the distance $bi$, between corresponding points of consecutive teeth, is called the *circular pitch* (usually denoted by $P$).

Knowing the pitch and the number ($N$) of teeth, the radius of the pitch circle will equal $P \times N \div 2\pi$.

As one inch pitch and twenty teeth are taken as data for the illustration, we have $CN = 3''.18 +$.

Fig. 422.




The other proportions are also expressed in terms of the pitch, a frequently-used system therefor being indicated on the figure.

If $i$ is a point through which a tooth outline is to pass, draw $Ci$, and on it as a diameter describe the semi-circumference $Csi$. An arc from centre $i$, with a radius of one-fourth $Ci$, will give the centre $s$ of the outline $hij$.

Draw the "circle of centres" through $s$, from centre $C$. Then with $si$ in the dividers, and from centre $f$ find $q$, which use in turn for arc $gfe$, and so continue.

*The width of rim*, $vw$, is often made, by a "shop" rule, equal to three-fourths the pitch. Reuleaux gives for it the following formula: $vw = 0.4\,P + .12$.

*Diametral pitch* is very frequently used instead of circular pitch, and is simply the number of teeth per inch of pitch-circle diameter.

658. *Helical Springs.* Draw first (Fig. 423) a central helix $aefm..T$, as follows: Divide $aa$,—

which is the *pitch*, or rise in one turn — into any number of equal parts, and the semi-circumference $AEM$ into half as many equal divisions; then each point marked with a capital on the half plan gives two elevations (denoted by the same letter small) by a process which is self-evident.

Fig. 423.

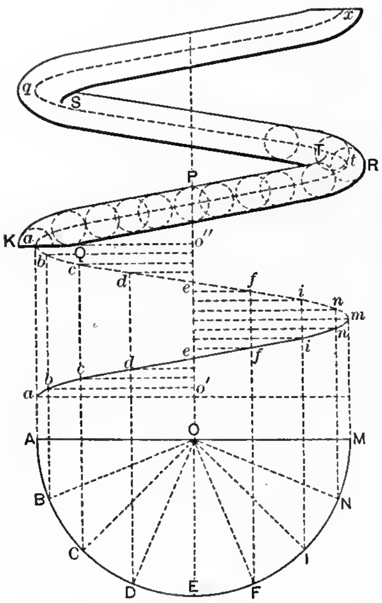
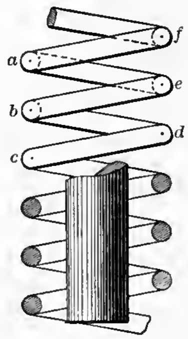
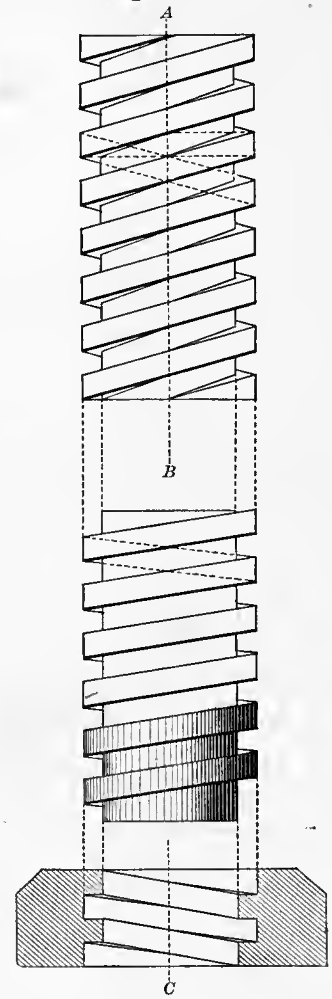



Fig. 424.




*If the spring is circular in cross-section* draw a series of circles having centres on the helix, and whose diameters equal that of the spring; then the outlines of the spring will be curves that are tangent to the circles.

Fig. 425.



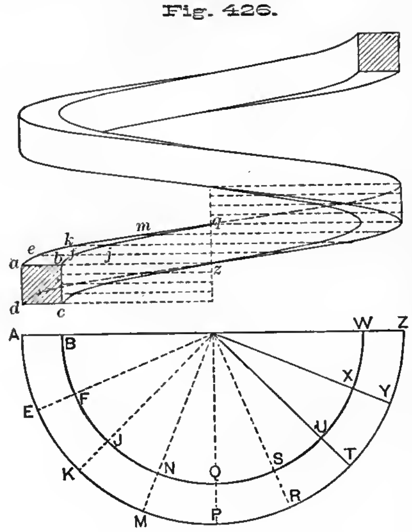

*If the spring be small* the curvature of the helix may be ignored, and a series of parallel straight lines employed instead, drawn tangent to circular arcs as in Fig. 424.

The upper half of the figure gives the method of construction, while the lower shows the spring in section, and surrounding a solid cylindrical core.

659. *Springs of rectangular cross-section.* Fig. 426 shows a spring of this description, formed by moving the rectangle $a\,b\,c\,d$ helically, each point describing a helix which can be constructed as described in the last article.

When any considerable number of turns of the same helix has to be drawn, it will save time if the draughtsman will shape a strip of pear-wood into a *templet*, i. e., a piece whose outline conforms to a line to be drawn or an edge to be cut, using it then as a curved ruler to guide his pen. This is the preferable method for all large work.

Fig. 426.




660. *Square-threaded screws.* If instead of spirally twisting a rectangular bar the same kind of surface be cut upon a cylinder of wood or metal, we shall have a square-threaded screw. This is illustrated by the upper part of Fig. 425, and its construction is self-evident after what has preceded. On a larger scale the curvature of the helices would have to be indicated.

The upper view is an elevation of a small *double-threaded* square screw, generated by winding two equal rectangles simultaneously around the axis.

The central figure is an elevation of a *single-threaded* screw. The lower figure is a sectional view of the nut for the single-threaded screw, and evidently presents a surface identical with that of the back half of the screw which fits it.

661. *Triangular-threaded screws.—United States Standard.*—The proportions devised by Mr. William Sellers of Philadelphia have been so generally adopted as to be known as the United States Standard. They are given in the table on the next page.

Fig. 427.

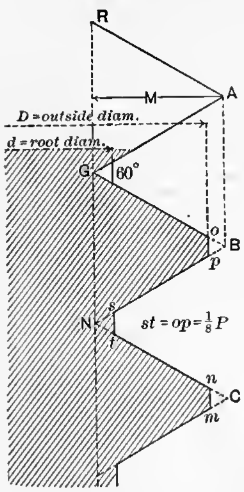



Fig. 428.

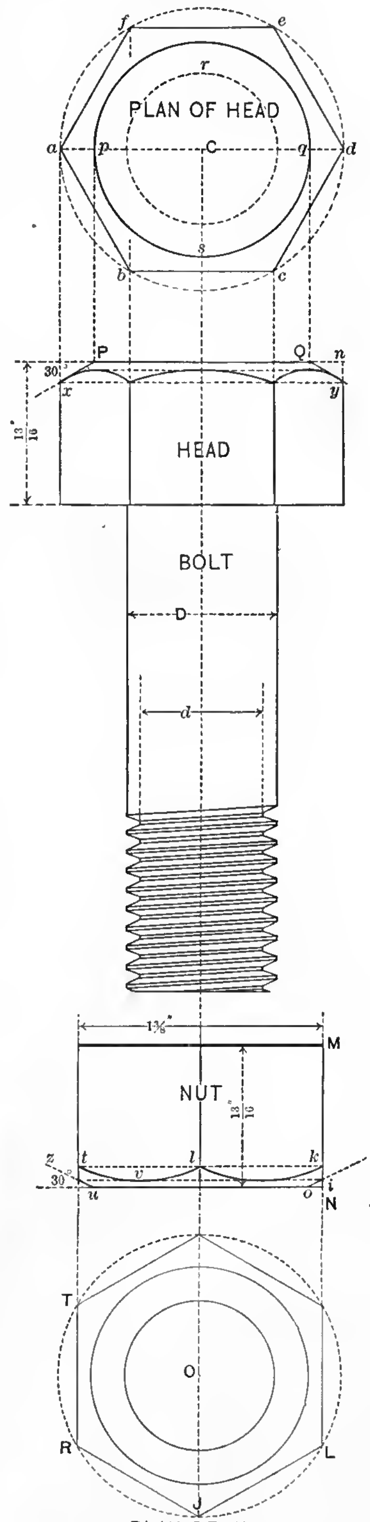



Fig. 427 shows a section of the Sellers screw. It is blunt on the thread, and also at the root. The part $o\,p\,B$ which is removed from the point may be regarded as filled in at $N\,s\,t$. $A\,B$ being the pitch ($P$), the widths $o\,p$, $s\,t$, are each one-eighth of $P$.

With $N$ equal to the number of threads per inch, and $D$ the outside diameter of the screw or bolt, the value of $d$—the diameter at the root—may be obtained from the formula $d = D - (1.299 \div N)$.

Other proportions are as follows: *The pitch* is equal to $0.24\sqrt{D+0.625} - 0.175$. *The depth of thread* equals $0.65\,P$. For bolts and nuts, whether hexagonal or square, the "width across flats," or shortest distance between parallel faces, equals $1.5\,D$, plus one-eighth of an inch for rough or unfinished surfaces, or plus one-sixteenth of an inch for "finished," i. e., machined or filed to smoothness.

The *depth of nut* equals the diameter of the bolt, for "rough" work. Tables should be consulted for the proportions of finished pieces.

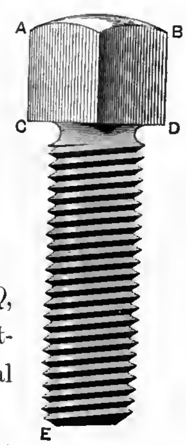
Fig. 428 is a drawing, to reduced scale, of a finished $\frac{7}{8}''$ bolt. The elevations show a bevel or chamfer, such as is usually given to a finished bolt or nut. On the plans this is indicated by the circles of diameter $p\,q$, the latter usually a little more than three-fourths of the diameter $a\,d$.

To draw the lines resulting from chamfering proceed thus: On a view showing "width across flats," as that of the nut, draw the chamfer lines $z\,u$, $o\,i$, at 30° to the top, and cutting off the desired amount. Draw circles on the plans, with diameter equal to $u\,o$. Project $p$ and $q$ to $P$ and $Q$ and draw $P\,x$ and $Q\,y$ at 30° to the top. Make $N\,k$ on the nut equal to $n\,y$ on the head. On the latter draw a parallel to $P\,Q$, and as far from it as $o\,u$ is from $v\,i$. The arcs limiting the plane faces have their centres found by "trial and error," three points of each curve being known.

Fig. 429.

Fig. 430.

Fig. 431.

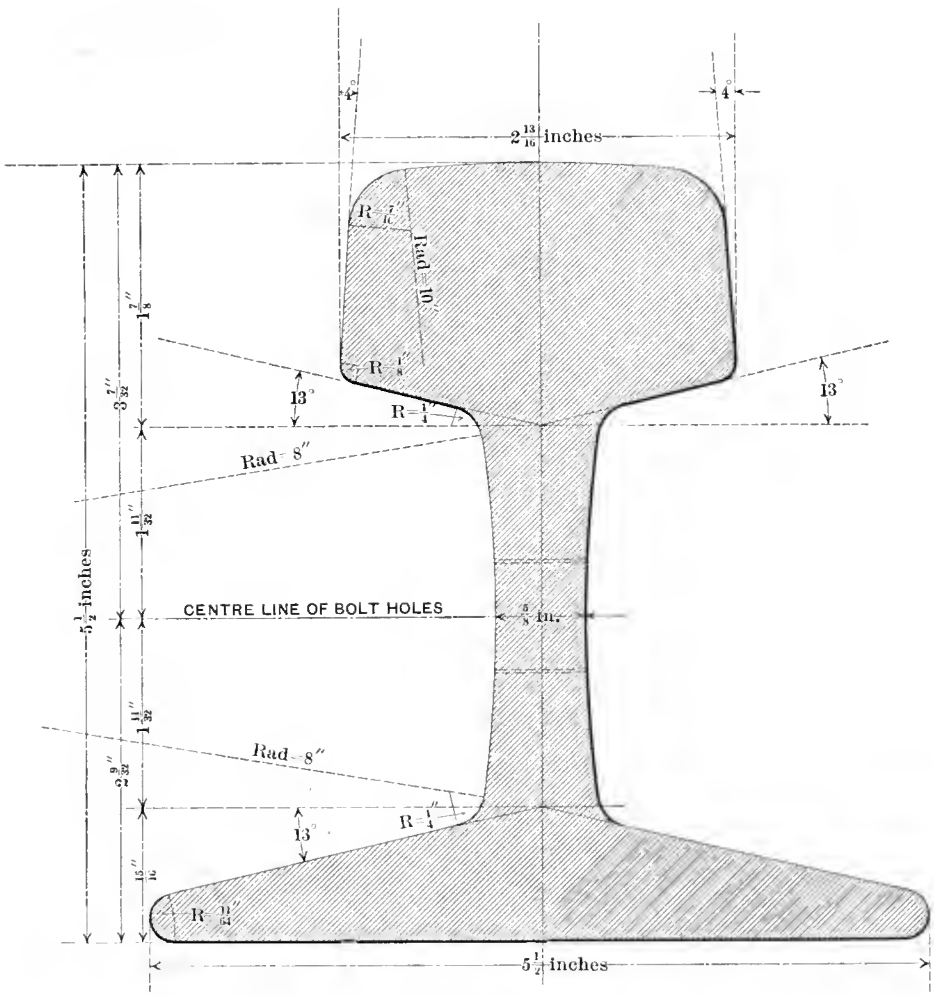



When drawn to a small scale screws may be represented by either of the conventional methods illustrated by Figs. 429, 430 and 431.

## DIMENSIONS OF BOLTS AND NUTS, UNITED STATES STANDARD (SELLERS SYSTEM)

| Proportions of Bolt | | | | | Dimensions of Nuts Rough and Finished | | | | | | Dimensions of Bolt Heads Rough and Finished | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Outside Diam. D | At Root of Thread | | N = Number of Threads per inch. | Width (f) of Flat. (60°) | Across Flats | | Across Corners (hexagon) | Across Corners (square) | Depth (h) of Nut | | Across Flats | | Across Corners (hexagon) | Across Corners (square) | Depth (h) of Head | |
| | Diam. | Area | | | R | F | R | R | R | F | R | F | R | R | R | F |
| $\frac{1}{4}$ | .185 | .026 | 20 | .0062 | $\frac{1}{2}$ | $\frac{7}{16}$ | $\frac{37}{64}$ | $\frac{7}{10}$ | $\frac{1}{4}$ | $\frac{3}{16}$ | $\frac{1}{2}$ | $\frac{7}{16}$ | $\frac{37}{64}$ | $\frac{7}{10}$ | $\frac{1}{4}$ | $\frac{3}{16}$ |
| $\frac{5}{16}$ | .240 | .045 | 18 | .0074 | $\frac{19}{32}$ | $\frac{17}{32}$ | $\frac{11}{16}$ | $\frac{5}{6}$ | $\frac{5}{16}$ | $\frac{1}{4}$ | $\frac{19}{32}$ | $\frac{17}{32}$ | $\frac{11}{16}$ | $\frac{5}{6}$ | $\frac{19}{64}$ | $\frac{1}{4}$ |
| $\frac{3}{8}$ | .294 | .067 | 16 | .0078 | $\frac{11}{16}$ | $\frac{5}{8}$ | $\frac{51}{64}$ | $\frac{63}{64}$ | $\frac{3}{8}$ | $\frac{5}{16}$ | $\frac{11}{16}$ | $\frac{5}{8}$ | $\frac{51}{64}$ | $\frac{63}{64}$ | $\frac{11}{32}$ | $\frac{5}{16}$ |
| $\frac{7}{16}$ | .344 | .092 | 14 | .0089 | $\frac{25}{32}$ | $\frac{23}{32}$ | $\frac{9}{10}$ | $1\frac{7}{64}$ | $\frac{7}{16}$ | $\frac{3}{8}$ | $\frac{25}{32}$ | $\frac{23}{32}$ | $\frac{9}{10}$ | $1\frac{7}{64}$ | $\frac{25}{64}$ | $\frac{3}{8}$ |
| $\frac{1}{2}$ | .400 | .125 | 13 | .0096 | $\frac{7}{8}$ | $\frac{13}{16}$ | 1 | $1\frac{15}{64}$ | $\frac{1}{2}$ | $\frac{7}{16}$ | $\frac{7}{8}$ | $\frac{13}{16}$ | 1 | $1\frac{15}{64}$ | $\frac{7}{16}$ | $\frac{7}{16}$ |
| $\frac{9}{16}$ | .454 | .161 | 12 | .0104 | $\frac{31}{32}$ | $\frac{29}{32}$ | $1\frac{1}{8}$ | $1\frac{23}{64}$ | $\frac{9}{16}$ | $\frac{1}{2}$ | $\frac{31}{32}$ | $\frac{29}{32}$ | $1\frac{1}{8}$ | $1\frac{23}{64}$ | $\frac{31}{64}$ | $\frac{1}{2}$ |
| $\frac{5}{8}$ | .507 | .201 | 11 | .0113 | $1\frac{1}{16}$ | 1 | $1\frac{7}{32}$ | $1\frac{1}{2}$ | $\frac{5}{8}$ | $\frac{9}{16}$ | $1\frac{1}{16}$ | 1 | $1\frac{7}{32}$ | $1\frac{1}{2}$ | $\frac{17}{32}$ | $\frac{9}{16}$ |
| $\frac{3}{4}$ | .620 | .301 | 10 | .0125 | $1\frac{1}{4}$ | $1\frac{3}{16}$ | $1\frac{7}{16}$ | $1\frac{49}{64}$ | $\frac{3}{4}$ | $\frac{11}{16}$ | $1\frac{1}{4}$ | $1\frac{3}{16}$ | $1\frac{7}{16}$ | $1\frac{49}{64}$ | $\frac{5}{8}$ | $\frac{11}{16}$ |
| $\frac{7}{8}$ | .731 | .419 | 9 | .0138 | $1\frac{7}{16}$ | $1\frac{3}{8}$ | $1\frac{21}{32}$ | $2\frac{1}{32}$ | $\frac{7}{8}$ | $\frac{13}{16}$ | $1\frac{7}{16}$ | $1\frac{3}{8}$ | $1\frac{21}{32}$ | $2\frac{1}{32}$ | $\frac{23}{32}$ | $\frac{13}{16}$ |
| 1 | .837 | .55 | 8 | .0156 | $1\frac{5}{8}$ | $1\frac{9}{16}$ | $1\frac{7}{8}$ | $2\frac{19}{64}$ | 1 | $\frac{15}{16}$ | $1\frac{5}{8}$ | $1\frac{9}{16}$ | $1\frac{7}{8}$ | $2\frac{19}{64}$ | $\frac{13}{16}$ | $\frac{15}{16}$ |
| $1\frac{1}{8}$ | .940 | .693 | 7 | .0178 | $1\frac{13}{16}$ | $1\frac{3}{4}$ | $2\frac{3}{32}$ | $2\frac{9}{16}$ | $1\frac{1}{8}$ | $1\frac{1}{16}$ | $1\frac{13}{16}$ | $1\frac{3}{4}$ | $2\frac{3}{32}$ | $2\frac{9}{16}$ | $\frac{29}{32}$ | $1\frac{1}{16}$ |
| $1\frac{1}{4}$ | 1.065 | .89 | 7 | .0178 | 2 | $1\frac{15}{16}$ | $2\frac{5}{16}$ | $2\frac{53}{64}$ | $1\frac{1}{4}$ | $1\frac{3}{16}$ | 2 | $1\frac{15}{16}$ | $2\frac{5}{16}$ | $2\frac{53}{64}$ | 1 | $1\frac{3}{16}$ |
| $1\frac{3}{8}$ | 1.160 | 1.056 | 6 | .0208 | $2\frac{3}{16}$ | $2\frac{1}{8}$ | $2\frac{17}{32}$ | $3\frac{3}{32}$ | $1\frac{3}{8}$ | $1\frac{5}{16}$ | $2\frac{3}{16}$ | $2\frac{1}{8}$ | $2\frac{17}{32}$ | $3\frac{3}{32}$ | $1\frac{3}{32}$ | $1\frac{5}{16}$ |
| $1\frac{1}{2}$ | 1.284 | 1.294 | 6 | .0208 | $2\frac{3}{8}$ | $2\frac{5}{16}$ | $2\frac{3}{4}$ | $3\frac{23}{64}$ | $1\frac{1}{2}$ | $1\frac{7}{16}$ | $2\frac{3}{8}$ | $2\frac{5}{16}$ | $2\frac{3}{4}$ | $3\frac{23}{64}$ | $1\frac{3}{16}$ | $1\frac{7}{16}$ |
| $1\frac{5}{8}$ | 1.389 | 1.515 | $5\frac{1}{2}$ | .0227 | $2\frac{9}{16}$ | $2\frac{1}{2}$ | $2\frac{31}{32}$ | $3\frac{5}{8}$ | $1\frac{5}{8}$ | $1\frac{9}{16}$ | $2\frac{9}{16}$ | $2\frac{1}{2}$ | $2\frac{31}{32}$ | $3\frac{5}{8}$ | $1\frac{9}{32}$ | $1\frac{9}{16}$ |
| $1\frac{3}{4}$ | 1.491 | 1.746 | 5 | .0250 | $2\frac{3}{4}$ | $2\frac{11}{16}$ | $3\frac{3}{16}$ | $3\frac{57}{64}$ | $1\frac{3}{4}$ | $1\frac{11}{16}$ | $2\frac{3}{4}$ | $2\frac{11}{16}$ | $3\frac{3}{16}$ | $3\frac{57}{64}$ | $1\frac{3}{8}$ | $1\frac{11}{16}$ |
| $1\frac{7}{8}$ | 1.616 | 2.051 | 5 | .0250 | $2\frac{15}{16}$ | $2\frac{7}{8}$ | $3\frac{13}{32}$ | $4\frac{5}{32}$ | $1\frac{7}{8}$ | $1\frac{13}{16}$ | $2\frac{15}{16}$ | $2\frac{7}{8}$ | $3\frac{13}{32}$ | $4\frac{5}{32}$ | $1\frac{15}{32}$ | $1\frac{13}{16}$ |
| 2 | 1.712 | 3.301 | $4\frac{1}{2}$ | .0277 | $3\frac{1}{8}$ | $3\frac{1}{16}$ | $3\frac{5}{8}$ | $4\frac{27}{64}$ | 2 | $1\frac{15}{16}$ | $3\frac{1}{8}$ | $3\frac{1}{16}$ | $3\frac{5}{8}$ | $4\frac{27}{64}$ | $1\frac{9}{16}$ | $1\frac{15}{16}$ |
| $2\frac{1}{4}$ | 1.962 | 3.023 | $4\frac{1}{2}$ | .0277 | $3\frac{1}{2}$ | $3\frac{7}{16}$ | $4\frac{1}{16}$ | $4\frac{61}{64}$ | $2\frac{1}{4}$ | $2\frac{3}{16}$ | $3\frac{1}{2}$ | $3\frac{7}{16}$ | $4\frac{1}{16}$ | $4\frac{61}{64}$ | $1\frac{3}{4}$ | $2\frac{3}{16}$ |
| $2\frac{1}{2}$ | 2.176 | 3.718 | 4 | .0312 | $3\frac{7}{8}$ | $3\frac{13}{16}$ | $4\frac{1}{2}$ | $5\frac{31}{64}$ | $2\frac{1}{2}$ | $2\frac{7}{16}$ | $3\frac{7}{8}$ | $3\frac{13}{16}$ | $4\frac{1}{2}$ | $5\frac{31}{64}$ | $1\frac{15}{16}$ | $2\frac{7}{16}$ |
| $2\frac{3}{4}$ | 2.426 | 4.622 | 4 | .0312 | $4\frac{1}{4}$ | $4\frac{3}{16}$ | $4\frac{29}{32}$ | 6 | $2\frac{3}{4}$ | $2\frac{11}{16}$ | $4\frac{1}{4}$ | $4\frac{3}{16}$ | $4\frac{29}{32}$ | 6 | $2\frac{1}{8}$ | $2\frac{11}{16}$ |
| 3 | 2.629 | 5.428 | $3\frac{1}{2}$ | .0357 | $4\frac{5}{8}$ | $4\frac{9}{16}$ | $5\frac{3}{8}$ | $6\frac{17}{32}$ | 3 | $2\frac{15}{16}$ | $4\frac{5}{8}$ | $4\frac{9}{16}$ | $5\frac{3}{8}$ | $6\frac{17}{32}$ | $2\frac{5}{16}$ | $2\frac{15}{16}$ |
| $3\frac{1}{4}$ | 2.879 | 6.509 | $3\frac{1}{2}$ | .0357 | 5 | $4\frac{15}{16}$ | $5\frac{13}{16}$ | $7\frac{1}{16}$ | $3\frac{1}{4}$ | $3\frac{3}{16}$ | 5 | $4\frac{15}{16}$ | $5\frac{13}{16}$ | $7\frac{1}{16}$ | $2\frac{1}{2}$ | $3\frac{3}{16}$ |
| $3\frac{1}{2}$ | 3.100 | 7.547 | $3\frac{1}{4}$ | .0384 | $5\frac{3}{8}$ | $5\frac{5}{16}$ | $6\frac{7}{64}$ | $7\frac{39}{64}$ | $3\frac{1}{2}$ | $3\frac{7}{16}$ | $5\frac{3}{8}$ | $5\frac{5}{16}$ | $6\frac{7}{64}$ | $7\frac{39}{64}$ | $2\frac{11}{16}$ | $3\frac{7}{16}$ |
| $3\frac{3}{4}$ | 3.317 | 8.641 | 3 | .0413 | $5\frac{3}{4}$ | $5\frac{11}{16}$ | $6\frac{21}{32}$ | $8\frac{1}{8}$ | $3\frac{3}{4}$ | $3\frac{13}{16}$ | $5\frac{3}{4}$ | $5\frac{11}{16}$ | $6\frac{21}{32}$ | $8\frac{1}{8}$ | $2\frac{7}{8}$ | $3\frac{13}{16}$ |
| 4 | 3.567 | 9.993 | 3 | .0413 | $6\frac{1}{8}$ | $6\frac{1}{16}$ | $7\frac{3}{32}$ | $8\frac{41}{64}$ | 4 | $3\frac{15}{16}$ | $6\frac{1}{8}$ | $6\frac{1}{16}$ | $7\frac{3}{32}$ | $8\frac{41}{64}$ | $3\frac{1}{16}$ | $3\frac{15}{16}$ |
| $4\frac{1}{4}$ | 3.798 | 11.329 | $2\frac{7}{8}$ | .0435 | $6\frac{1}{2}$ | $6\frac{7}{16}$ | $7\frac{9}{16}$ | $9\frac{3}{16}$ | $4\frac{1}{4}$ | $4\frac{3}{16}$ | $6\frac{1}{2}$ | $6\frac{7}{16}$ | $7\frac{9}{16}$ | $9\frac{3}{16}$ | $3\frac{1}{4}$ | $4\frac{3}{16}$ |
| $4\frac{1}{2}$ | 4.028 | 12.742 | $2\frac{3}{4}$ | .0454 | $6\frac{7}{8}$ | $6\frac{13}{16}$ | $7\frac{31}{32}$ | $9\frac{3}{4}$ | $4\frac{1}{2}$ | $4\frac{7}{16}$ | $6\frac{7}{8}$ | $6\frac{13}{16}$ | $7\frac{31}{32}$ | $9\frac{3}{4}$ | $3\frac{7}{16}$ | $4\frac{7}{16}$ |
| $4\frac{3}{4}$ | 4.256 | 14.226 | $2\frac{5}{8}$ | .0476 | $7\frac{1}{4}$ | $7\frac{3}{16}$ | $8\frac{13}{32}$ | $10\frac{1}{4}$ | $4\frac{3}{4}$ | $4\frac{11}{16}$ | $7\frac{1}{4}$ | $7\frac{3}{16}$ | $8\frac{13}{32}$ | $10\frac{1}{4}$ | $3\frac{5}{8}$ | $4\frac{11}{16}$ |
| 5 | 4.480 | 15.763 | $2\frac{1}{2}$ | .0500 | $7\frac{5}{8}$ | $7\frac{9}{16}$ | $8\frac{27}{32}$ | $10\frac{49}{64}$ | 5 | $4\frac{15}{16}$ | $7\frac{5}{8}$ | $7\frac{9}{16}$ | $8\frac{27}{32}$ | $10\frac{49}{64}$ | $3\frac{13}{16}$ | $4\frac{15}{16}$ |
| $5\frac{1}{4}$ | 4.730 | 17.570 | $2\frac{1}{2}$ | .0500 | 8 | $7\frac{15}{16}$ | $9\frac{9}{32}$ | $11\frac{23}{64}$ | $5\frac{1}{4}$ | $5\frac{3}{16}$ | 8 | $7\frac{15}{16}$ | $9\frac{9}{32}$ | $11\frac{23}{64}$ | 4 | $5\frac{3}{16}$ |
| $5\frac{1}{2}$ | 4.953 | 19.267 | $2\frac{3}{8}$ | .0526 | $8\frac{3}{8}$ | $8\frac{5}{16}$ | $9\frac{23}{32}$ | $11\frac{7}{8}$ | $5\frac{1}{2}$ | $5\frac{7}{16}$ | $8\frac{3}{8}$ | $8\frac{5}{16}$ | $9\frac{23}{32}$ | $11\frac{7}{8}$ | $4\frac{3}{16}$ | $5\frac{7}{16}$ |
| $5\frac{3}{4}$ | 5.203 | 21.261 | $2\frac{3}{8}$ | .0526 | $8\frac{3}{4}$ | $8\frac{11}{16}$ | $10\frac{5}{32}$ | $12\frac{3}{8}$ | $5\frac{3}{4}$ | $5\frac{11}{16}$ | $8\frac{3}{4}$ | $8\frac{11}{16}$ | $10\frac{5}{32}$ | $12\frac{3}{8}$ | $4\frac{3}{8}$ | $5\frac{11}{16}$ |
| 6 | 5.243 | 23.097 | $2\frac{1}{4}$ | .0555 | $9\frac{1}{8}$ | $9\frac{1}{16}$ | $10\frac{19}{32}$ | $12\frac{15}{16}$ | 6 | $5\frac{15}{16}$ | $9\frac{1}{8}$ | $9\frac{1}{16}$ | $10\frac{19}{32}$ | $12\frac{15}{16}$ | $4\frac{9}{16}$ | $5\frac{15}{16}$ |

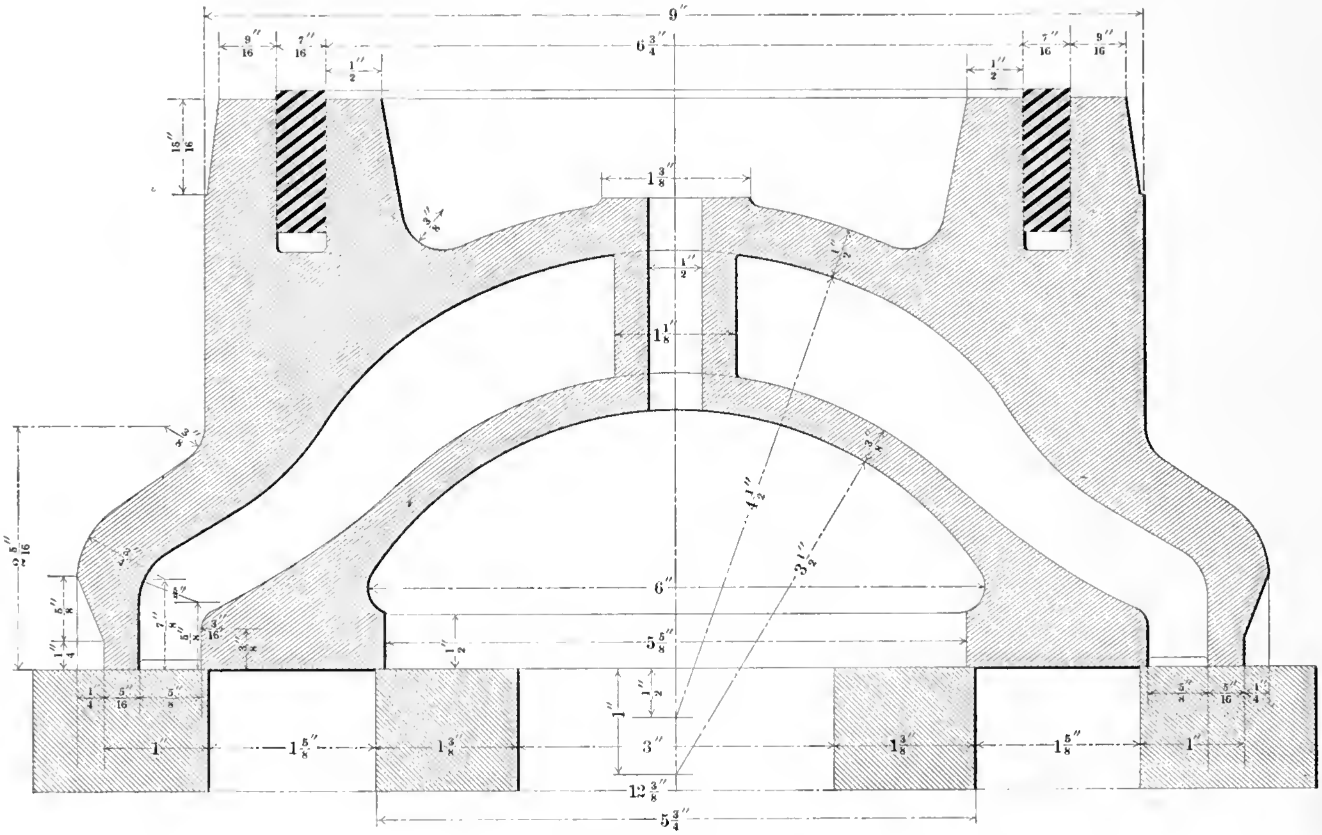
# APPENDIX




# P. R. R. STANDARD RAIL SECTION.

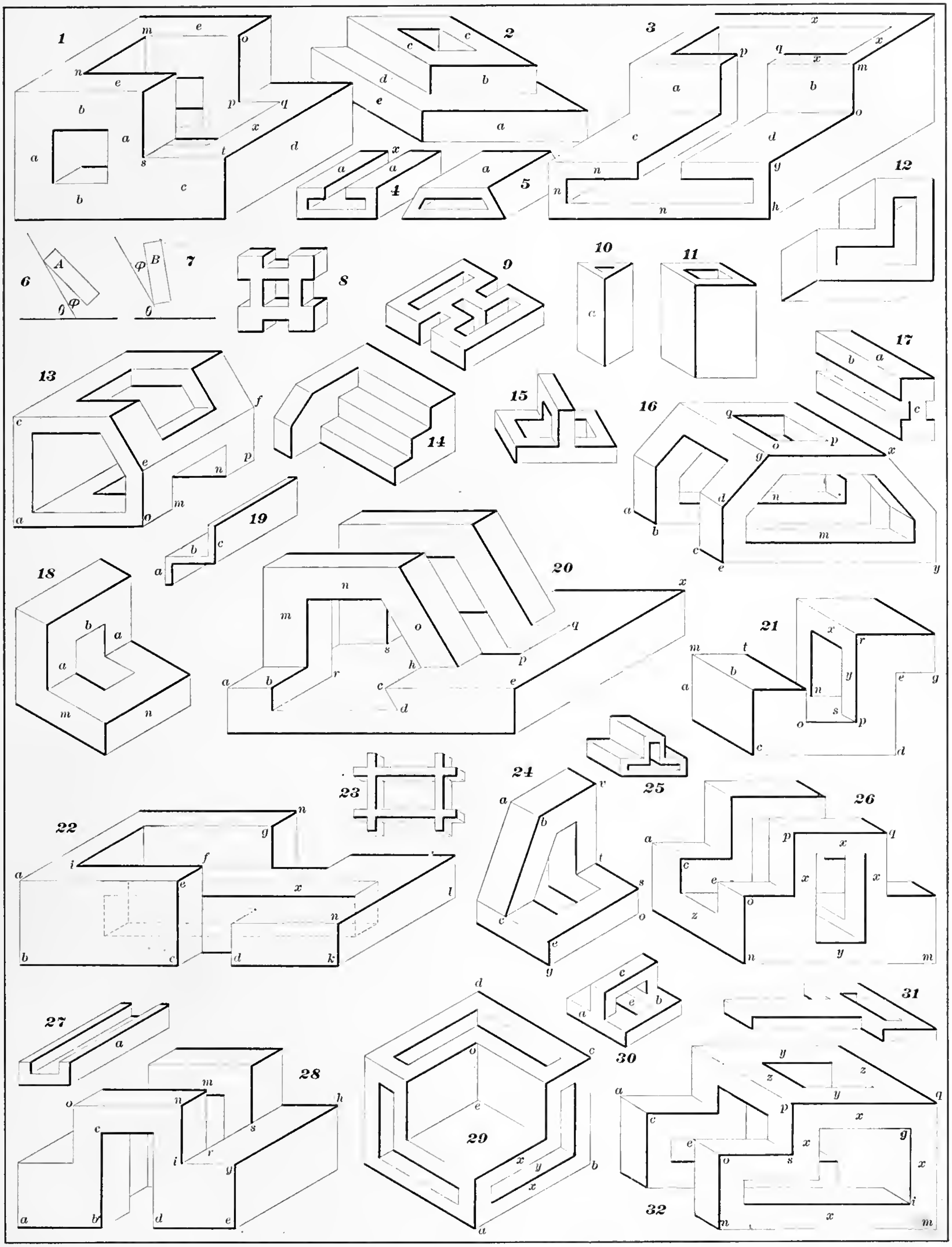
100 LBS. PER YARD.

Draw the above either full size or enlarged 50%. In either case draw section lines in Prussian blue, spacing not less than one-twentieth of an inch. Dimension lines, red. Dimensions and arrow heads, black. Lettering and numerals either in Extended Gothic or Reinhardt Gothic.




## ALLEN-RICHARDSON SLIDE VALVE.

Draw either full size or larger. Section lines in Prussian blue, one-twentieth of an inch apart. Dimension lines, red. Dimensions and arrow heads, black. Lettering and numerals either in Extended Gothic or Reinhardt Gothic.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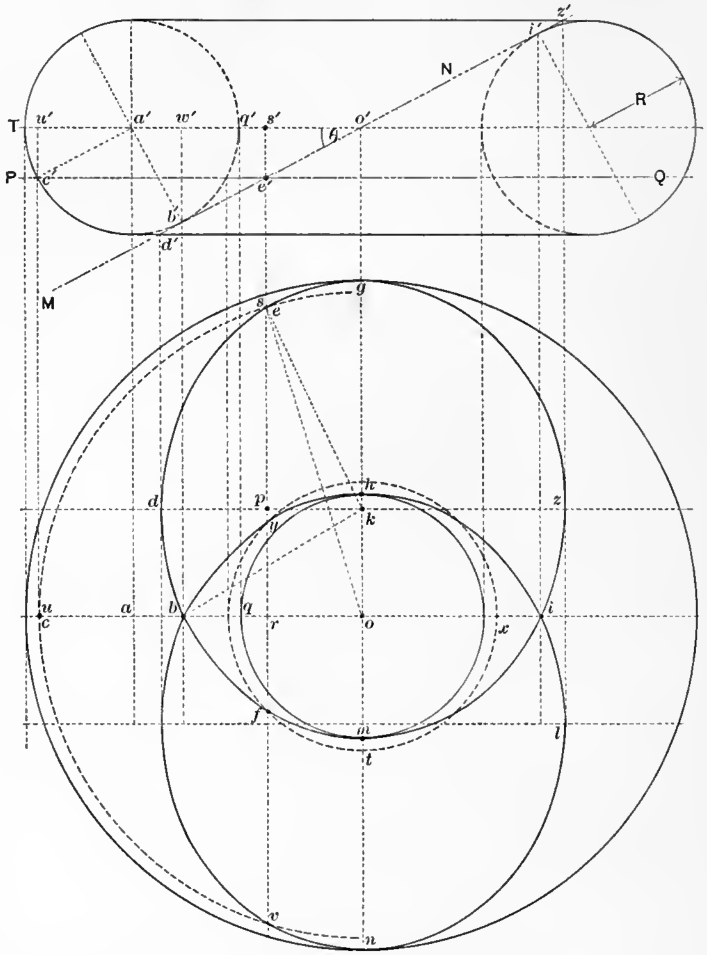
PROOF THAT A BI-TANGENT PLANE TO AN ANNULAR TORUS CUTS IT IN TWO EQUAL CIRCLES. (SEE ART. 113).




Let $d\,m\,z\,y$ and $n\,l\,h\,b$ be the plans of the curves of section, $d'z'$ their common elevation. The plane $MN$ cuts the equators of the surface at the points $g$, $h$, $m$, $n$; and if the sections are circles their diameters must obviously equal $g\,m$ or $h\,n$.

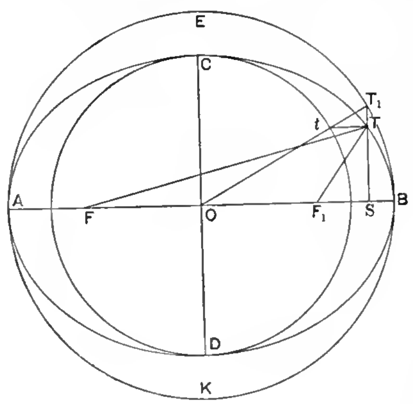
Take $g\,k$ equal to one-half $g\,m$. Let $R$ denote the radius of the generating circle of the torus. Then $g\,k = R + o\,h = a'o'$. We have also $o\,k = R$.

Assume any horizontal plane $PQ$, cutting the torus in the circles $e\,c\,v$ and $t\,x\,y$, and the plane $MN$ in the line $e\,v$, $e'$. This line gives $e$ and $f$ on one of the curves, $y$ and $v$ on the other. Their elevation is $e'$.

If $e\,d\,m$ is a circle we must have $e\,k = g\,k = a'o'$. Draw $e\,o$; make $k\,d$ parallel to $o\,a$, and $e\,r$ perpendicular to it. Then, as the difference of level of the points $k$ and $e$ is seen at $s'e'$, we will have $\sqrt{k\,e^2 + s'e'^2}$ for the true length of the line whose plan is $k\,e$; and $k\,e^2 + s'e'^2$ is to equal $g\,k^2$.

In the right triangle $s\,p\,k$ we have $k\,e^2 = p\,k^2 + s\,p^2$. Then $k\,e^2 + s'e'^2 = p\,k^2 + s\,p^2 + s'e'^2$. The second member becomes $a'o'^2$ by substitution and reduction. For $p\,k^2\,(= o's'^2)$ employ $[(s'e'^2.\,a'o'^2 - s'e'.^2\,R^2) \div R^2]$, derived from triangles $o's'e'$ and $o'a'b'$; and as $p\,s = r\,s - r\,p = r\,s - R$, we have $s\,p^2 = (r\,s - R)^2 = (\sqrt{o\,s^2 - o\,r^2} - R)^2 = (\sqrt{o'a'^2 + a'u'^2 - o's'^2} - R)^2$. $R$ and $a'u'$ disappear by using values derived from the triangle $a'u'c'$.

NOTE TO ART. 131, ON THE ELLIPSE AND ITS AUXILIARY CIRCLES.




The relation of $T$ to $t$ and $T_1$ is thus shown analytically: Representing lines by letters, let $O\,A\,(= O\,B = F\,C) = a$; $O\,C = b$; $O\,F = O\,F_1 = c$, a constant quantity; $O\,S = x$; $S\,T = y$; $S\,T_1 = y'$; $F\,T = \rho$; $F_1\,T = \rho'$. Then $\rho + \rho' = 2\,a = \sqrt{y^2 + (x + c)^2} + \sqrt{y^2 + (x - c)^2}$, which, after squaring, and substituting $b^2$ for $a^2 - c^2$, gives $b^2 x^2 + a^2 y^2 = a^2 b^2$, the well-known equation to the ellipse; written also $y^2 = \frac{b^2}{a^2}(a^2 - x^2) \ldots (1)$.

In the circle $A\,E\,B\,K$ we have $O\,T_1 = r = a = \sqrt{S\,T_1^2 + O\,S^2} = \sqrt{(y')^2 + x^2}$; whence $(y')^2 = a^2 - x^2 \ldots (2)$.

Dividing (1) by (2), remembering that $x$ is the same for both $T_1$ and $T$, we have $\frac{y^2}{(y')^2} = \frac{b^2}{a^2}$, whence $y : y' :: b : a$; that is, the ordinate of the ellipse is to the ordinate of the circle as the semi-conjugate axis is to the semi-transverse. But in the similar triangles $T_1 S\,O$, $T_1 T\,t$, we have $S\,T : S\,T_1 :: o\,t : O\,T_1$; that is, $y : y' :: b : a$, the relation just established otherwise for a point of an ellipse.

# THE NOMENCLATURE

# AND

# DOUBLE GENERATION OF TROCHOIDS.

## *THE NOMENCLATURE AND DOUBLE GENERATION OF TROCHOIDS.*

[The anomalies and inadequateness of the pre-existing nomenclature of trochoidal curves led to an attempt on the part of the writer to simplify the matter, and the following paper is, in substance, that presented upon the subject before the American Association for the Advancement of Science, in 1887. Two brief quotations from some of the communications to which it led will indicate the result.

From Prof. Francis Reuleaux, Director of the Royal Polytechnic Institution, Berlin:

"*I agree with pleasure to your discrimination of major, minor and medial hypotrochoids and will in future apply these novel designations.*"

From Prof. Richard A. Proctor, B.A., author of *Geometry of Cycloids*, etc.:

"*Your system seems complete and satisfactory. I was conscious that my own suggestions were but partially corrective of the manifest anomalies in former nomenclatures.*"

The final outcome of the investigation, as far as technical terms are concerned, appears on page 59, in a tabular arrangement suggested by that of Kennedy, and which is both a modification and an extension of his ingenious scheme. The property of double generation of trochoids, when the tracing-point is not on the circumference of the rolling circle, is even at present writing not treated by some authors of advanced text-books who nevertheless emphasize it for the epi-, hypo- and peri-*cycloid.* This fact, and the importance of the property both in itself and as leading to the solution of a vexed question, are my main reasons for introducing the paper here in nearly its original length; although to the student of mathematical tastes the original demonstration presented may prove to be not the least interesting feature of the investigation.

The demonstrations alone might have appeared in Chapter V—their rightful setting had this been merely a treatise on plane curves, but they would there have unduly lengthened an already large division of the work, while at that point their especial significance could not, for the same reason, have been sufficiently shown.]

That would be an ideal nomenclature in which, from the etymology of the terms chosen, so clear an idea could be obtained of that which is named as to largely anticipate definition, if not, indeed, actually to render it superfluous. This ideal, it need hardly be said, is seldom realized. As a rule we meet with but few self-explanatory terms, and the greater their lack of suggestiveness the greater the need of clear definition. Instances are not wanting of ill-chosen terms and even actual misnomers having become so generally adopted, in spite of an occasional protest, that we can scarcely expect to see them replaced by others more appropriate. Whether this be the case or not, we have a right to expect, especially in the exact sciences, and preëminently in Mathematics, such clearness and comprehensiveness of definition as to make ambiguity impossible. But in this we are frequently disappointed, and notably so in the class of curves we are to consider.

Toward the close of the seventeenth century the mechanician De la Hire gave the name of *Roulette*—or roll-traced curve—to the path of a point in the plane of a curve rolling upon any other curve as a base. This suggestive term has been generally adopted, and we may expect its complementary, and equally self-interpreting term, *Glissette*, to keep it company for all time.

By far the most interesting and important roulettes are those traced by points in the plane of a circle rolling upon another circle in the same plane, such curves having valuable practical applications in mechanism, while their geometrical properties have for centuries furnished an attractive field for investigation to mathematicians.

The terms *Cycloids* and *Trochoids* have been somewhat indiscriminately used as general names for this class of curves. As far as derivation is concerned they are equally appropriate, the former being from κύκλος, circle, and εἶδος, form; and the latter from τρόχος, wheel, and εἶδος. Preference has, however, been given to the term *Trochoids* by several recent writers on mathematics or mechanism, among them Prof. R. H. Thurston and Prof. De Volson Wood; also Prof. A. B. W. Kennedy of England, the translator of Reuleaux' *Theoretische Kinematik*, in which these curves figure so largely as centroids. Adopting it for the sake of aiding in establishing uniformity in nomenclature I give the following definition:

*If two circles are tangent, either externally or internally, and while one of them remains fixed the other rolls upon it without sliding, the curve described by any point on a radius of the rolling circle, or on a radius produced, will be a Trochoid.*

Of these curves the most interesting, both historically and for its mathematical properties, is the *cycloid*, with which all are familiar as the path of a point on the circumference of a circle which rolls upon a straight line, i. e., the circle of infinite radius.

The term "cycloid" alone, for the locus described, is almost universally employed, although it is occasionally qualified by the adjectives *right* or *common.*

Of almost equally general acceptation, although frequently inappropriate, are the adjectives *curtate* and *prolate*, to indicate trochoidal curves traced by points respectively *without* and *within* the circumference of the rolling circle (or *generator* as it will hereafter be termed) whether it roll upon a circle of finite or infinite radius.

As distinguished from curtate and prolate forms all the other trochoids are frequently called *common.*

Should the fixed circle (called either the *base* or *director*) have an infinite radius, or, in other words, be a straight line, the curtate curve is called by some *the curtate cycloid;* by others the *curtate trochoid;* and similarly for the prolate forms. Since uniformity is desirable I have adopted the terms which seem to have in their favor the greater number of the authorities consulted, viz., curtate and prolate *trochoid.* It should also be further stated here, with reference to this word "trochoid," that it is usually the termination of the name of every curtate and prolate form of trochoidal curve, the termination *cycloid* indicating that the tracing point is *on* the circumference of the generator.

With the base a straight line the curtate form consists of a series of loops, while the prolate forms are sinuous, like a wave line; and the same is frequently true when the base is a circle of finite radius; hence the suggestion of Prof. Clifford that the terms *looped* and *wavy* be employed instead of *curtate* and *prolate.* But we shall see, as we proceed, that they would not be of universal applicability, and that, except with a straight line director, both curtate and prolate curves may be, in form, looped, wavy, or neither. And we would all agree with Prof. Kennedy that as substitutes for these terms "Prof. Cayley's *kru-nodal* and *ac-nodal* hardly seem adapted for popular use." It is therefore futile to attempt to secure a nomenclature which shall, throughout, suggest both the *form* of the locus and the *mode of its construction*, and we must rest content if we completely attain the latter desideratum.

We have next to consider the trochoids traced during the rolling of a circle upon another circle of finite radius. At this point we find inadequacy in nomenclature, and definitions involving singular anomalies. The earlier definitions have been summarized as follows by Prof. R. A. Proctor, in his valuable *Geometry of Cycloids:*—

"The $\left\{\begin{matrix}\text{epicycloid}\\ \text{hypocycloid}\end{matrix}\right\}$ is the curve traced out by a point in the circumference of a circle which rolls without sliding on a fixed circle in the same plane, the two circles being in $\left\{\begin{matrix}\text{external}\\ \text{internal}\end{matrix}\right\}$ contact."

As a specific example of this class of definition I quote the following from a more recent writer:—"If the generating circle rolls on the circumference of a fixed circle, instead of on a fixed line, the curve generated is called an epicycloid if the rolling circle and the fixed circle are tangent externally, a hypocycloid if they are tangent internally." (Byerly, *Differential Calculus*, 1880.)

In accordance with the foregoing definitions every epicycloid is also a hypocycloid, while only some hypocycloids are epicycloids. Salmon (*Higher Plane Curves*, 1879) makes the following explicit statement on this point:—"The hypocycloid, when the radius of the moving circle is greater than that of the fixed circle, may also be generated as an epicycloid."

To avoid any anomaly Prof. Proctor has presented the following unambiguous definition:—

"An $\left\{\begin{matrix}\text{epicycloid}\\ \text{hypocycloid}\end{matrix}\right\}$ is the curve traced out by a point on the circumference of a circle which rolls without sliding on a fixed circle in the same plane, the rolling circle touching the $\left\{\begin{matrix}\text{outside}\\ \text{inside}\end{matrix}\right\}$ of the fixed circle."

This certainly does away with all confusion between the *epi-* and *hypo*-curves, but we shall find it inadequate to enable us, clearly, to make certain desirable distinctions.

By some writers the term *external epicycloid* is used when the generator and director are tangent externally, and, similarly, *internal epicycloid* when the contact is internal and the larger circle is rolling. Instead of *internal epicycloid* we often find *external hypocycloid* used. It will be sufficient, with regard to it, to quote the following from Prof. Proctor:—"It has hitherto been usual to define it (the hypocycloid) as the curve obtained when either the convexity of the rolling circle touches the concavity of the fixed circle, or the concavity of the rolling circle touches the convexity of the fixed circle. There is a manifest want of symmetry in the resulting classification, seeing that while every epicycloid is thus regarded as an external hypocycloid, no hypocycloid can be regarded as an internal epicycloid. Moreover, an external hypocycloid is in reality an anomaly, for the prefix 'hypo,' used in relation to a closed figure like the fixed circle, implies interiorness."

To avoid the confusion which it is evident from the foregoing has existed, and at the same time to conform to that principle which is always a safe one and never more important than in nomenclature, viz., not to use two words where one will suffice, I prefer reserving the term "epicycloid" for the case of external tangency, and substituting the more recently suggested name *pericycloid* for both "internal epicycloid" and "external hypocycloid." The curtate and prolate forms would then be called *peritrochoids.* By the use of these names and those to be later presented we can easily make distinctions which, without them, would involve undue verbiage in some cases, and, in others, the use of the ambiguous or inappropriate terms to which exception is taken. And the necessity for such distinctions frequently arises, especially in the study of kinematics and machine design. Take, for example, problems like many in the work of Reuleaux already mentioned, relating to the relative motion of higher kinematic pairs of elements, the centroids being circular arcs and the point-paths trochoids. In such cases we are quite as much concerned with the relative position of the rolling and fixed circles as with the form of a point-path. In solving problems in gearing the same need has been felt of simple terms for the trochoidal profiles of the teeth, which should imply the method of their generation.

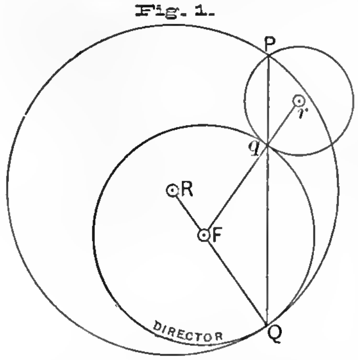
Although they have not, as yet, come into general use, the names pericycloid and peritrochoid appear in the more recent editions of Weisbach and Reuleaux, and will undoubtedly eventually meet with universal acceptance.

Yet strong objection has been made to the term "pericycloid" by no less an authority than the late eminent mathematician, Prof. W. K. Clifford, who nevertheless adopted the "peritrochoid." I quote the following from his *Elements of Dynamic:*—"Two circles may touch each other so that each is outside the other, or so that one includes the other. In the former case, if one circle rolls upon the other, the curves traced are called epicycloids and epitrochoids. In the latter case, if the inner circle roll on the outer, the curves are hypocycloids and hypotrochoids, but if the outer circle roll on the inner, the curves are epicycloids and peritrochoids. We do not want the name pericycloids, because, as will be seen, every pericycloid is also an epicycloid; but there are three distinct kinds of trochoidal curves." As it will later be shown that every peri-*trochoid* can also be generated as an epi-trochoid we can scarcely escape the conclusion that the name *peritrochoid* would also have been rejected by Prof. Clifford, had he been familiar with this property of double generation as belonging to the curtate and prolate forms as well. But it is this very property, possessed also by the *hypo*-trochoids, which necessitates a more extended nomenclature than that heretofore existing, and I am not aware that there has been any attempt to provide the nine terms essential to its completeness. These it is my principal object to present, and that they have not before been suggested I attribute to the fact that the double generation of curtate and prolate trochoidal curves does not seem to have been generally known, being entirely ignored in many treatises which make quite prominent the fact that it is a property of the epi- and hypo-*cycloids*, while, as far as I have seen, the only writer who mentions it proves it indirectly, by showing the identity of trochoids with epicyclics and establishing it for the latter.

As it is upon this peculiar and interesting feature that the nomenclature, as now extended, depends, the demonstrations necessary to establish it are next in order.

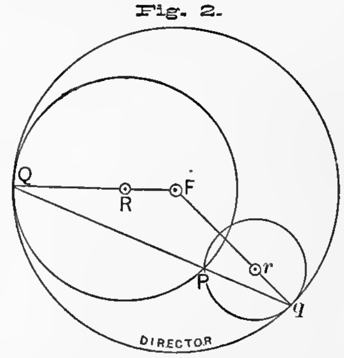
For the epi- and hypo-*cycloid* probably the simplest method of proof is that based upon the instantaneous centre, and which we may call a kinematic, as distinguished from a strictly geometrical, demonstration. It is as follows:—

Fig. 1.



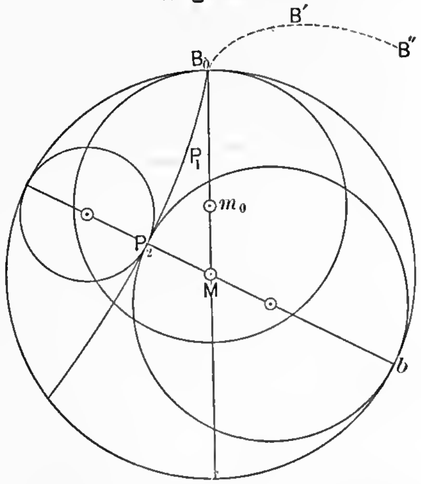

Let F (Figs. 1 and 2) be the centre of the fixed circle, and $r$ that of a rolling circle, the tracing point, P, being *on* the circumference of the latter. The point of contact, $q$, is—at the moment that the circles are in the relative position indicated—an instantaneous centre of rotation for every point in the plane of the rolling circle; the line P$q$, joining such point of contact with the tracing point, is therefore a normal to the trochoid that the point P is tracing. But if the normal P$q$ be produced to intersect the fixed circle in a second point, Q, it is evident that the same infinitesimal arc of the trochoid would be described with Q serving as instantaneous centre as when $q$ fulfilled that office. The point P will, therefore, evidently trace the same curve, whether it be considered as in the circumference of the circle $r$, or in that of a second and larger circle, R, tangent to the fixed circle at Q.

Fig. 2.



It is worth while, in this connection, to note what erroneous ideas with regard to these same loci were held by some writers as late as the middle of this century,—ideas whose falsity it would seem as if the most elementary geometrical construction would have exposed. Reuleaux instances the following statement made by Weissenborn in his *Cyclischen Kurven* (1856): "If the circle described about $m_0$ roll upon that described about M, and if the describing point, $B_0$, describe the curve $B_0 P_1 P_2$ as the inner circle rolls upon the arc $B_0 b$, then, evidently, if the smaller circle be fixed and the larger one rolled upon it in a direction opposite to that of the former rotation, the point of the great circle which at the beginning of the operation coincided with $B_0$ describes the same line $B_0 P_1 P_2$." The fallacy of this statement is to us, perhaps, in the light of what has preceded, a little more evident than Weissenborn's deduction; although, as Reuleaux says, "his 'evidently' expresses the usual notion, and the one which is suggested by a hasty pre-judgment of the case. In point of fact $B_0$ describes the pericycloid $B_0 B' B''$, which certainly differs sufficiently from the hypocycloid $B_0 P_1 P_2$."

Fig. 3.




We have next to consider the curtate and prolate epi-, hypo- and peri-trochoids.

As previously stated, I have seen no direct proof that they also possess the same property of double generation, but find that the kinematic method lends itself with equal readiness to its demonstration.

For the hypotrochoids, let R, Fig. 4, be the centre of the first rolling circle or generator, F that of the first director, and P the initial position of the tracing point. The initial point of tangency of generator and director is $m$. Let the generator roll over *any* arc of the director, as $m$Q. The centre R will then be found at $R_2$, and the tracing point P at $P_2$. The point of contact, Q, will then be the instantaneous centre of rotation for $P_2$, and $P_2 Q$ will, therefore, be a normal to the trochoid for that particular position of the tracing point.

The motion of P is evidently circular about R, while that of R is in a circle about F. The curve $P P_1 P_2 \ldots\ldots P_6$ is that portion of the hypotrochoid which is described while P describes an arc of 180° about R, the latter meanwhile moving through an arc of 108° about F, the ratio of the radii being 3 : 5.

Now while tracing the curve indicated the point P can be considered as rigidly connected with a second point, $\rho$, about which it also describes a circle, $\rho$ meanwhile (like R) describing a circle about F. Such a point may be found as follows:—Take any position of P, as $P_2$, and join it with the corresponding position of R, as $R_2$; also join $R_2$ to F. Let us then suppose $P_2 R$ and $R_2 F$ to be adjacent links of a four-link mechanism. Let the remaining links, $F\rho_2$ and $\rho_2 P_2$, be parallel and equal to $P_2 R_2$ and $R_2 F$ respectively. Taking F as the fixed point of the mechanism let us suppose $P_2$ moved toward it over the path $P_2 P_3 \ldots P_6$. Both $R_2$ and $\rho_2$ will evidently describe circular arcs about F; while the motion of $P_2$ with respect to $\rho_2$ will be in a circular arc of radius $\rho_2 P_2$. We may, therefore, with equal correctness, consider $\rho_2$ as the centre of a generator carrying the point $P_2$, and $\rho_2 F$ a new line of centres, intersected by the normal $P_2 Q$ in a second instantaneous centre, $q$, which, in strictest analogy with Q, divides the line of centres on which it lies into segments, $\rho_2 q$ and F$q$, which are the radii of the second generator and director respectively; $q$ being, like Q, the point of contact of the rolling and fixed circles for the instant that the tracing point is at $P_2$. The second generator and director, having $\rho_2 q$ and $q$F respectively for their radii, are represented in their initial positions, $\rho$ being the centre of the former, and $\mu$ the initial point of contact. The second generator rolls in the opposite direction to the first.

It is important to notice that whereas the tracing point is in the first case *within* the generator and therefore traces the curve as a *prolate* hypotrochoid, it is *without* the second generator and describes the same curve as a *curtate* hypotrochoid. If we now let R and F denote no longer the *centres*, but the *radii*, of the rolling and fixed circles, respectively, we have for the first generator and director $2R > F$, and for the second $2R < F$.

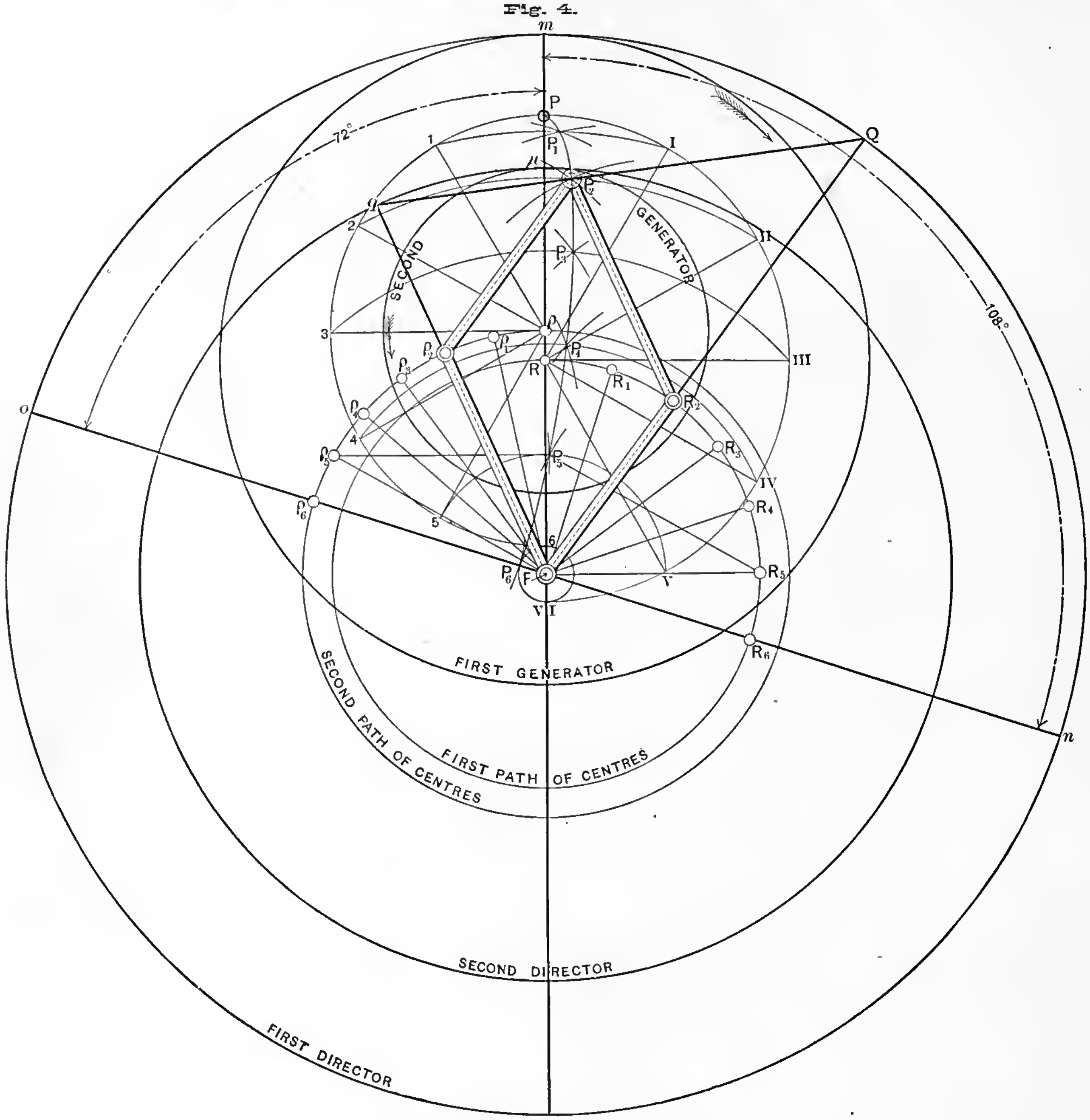
It occurred to me that a distinction could very easily be made between trochoids generated under these two opposite relations of radii, by using the simple and suggestive term *major hypotrochoid* when 2 R is greater than F, and *minor hypotrochoid* when the opposite relation prevails. We would then say that the preceding demonstration had established the identity of a major prolate with a minor curtate hypotrochoid.

Similarly the identity of major curtate and minor prolate forms could be shown.

If the tracing point were *on* the circumference of the generator the trochoids traced would be, by the new nomenclature, major and minor hypo-*cycloids.*

It is worth noticing that for both hypo-*cycloids* and hypo-*trochoids* the *centre* F is the same for both generations, and that the *radius* F is also constant for both generations of a hypo-*cycloid*, but variable for those of a hypo-*trochoid.*

Fig. 4.




DOUBLE GENERATION OF HYPOTROCHOIDS.

Having given the radii of generator and director for the construction of a hypo-trochoid, the method just illustrated will always give the lengths of the radii of the second rolling and fixed circles. The accuracy of the values thus obtained may be checked by simple formulae derived from the same figure, as follows:—

Radii being given for generation as a *major* hypotrochoid, to find corresponding values for the identical *minor* hypotrochoid.

Let $F_1$ denote the radius $FQ$ $[= Fm]$ of the first director.

" $F_2$ " " " $Fq$ $[= F\mu]$ " " second "

" $r$ " " " $R_2Q$ $[= Rm]$ " " first generator.

" $\rho$ " " " $\rho_2 q$ $[= \rho\mu]$ " " second "

" $tr$ " " *tracing radius* of the first generation, i. e., the distance $R_2P_2$ (or $RP$) of tracing point from centre of first generator.

Let $t\rho$ equal the second tracing radius $= \rho_2 P_2 = \rho P$.

From the similar triangles $QFq$ and $QR_2P_2$ we have $F_2 : F_1 :: tr : r$

whence $F_2 = \dfrac{F_1\,(tr)}{r}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (1)

also $\rho = F_2 - tr = \dfrac{F_1\,(tr)}{r} - tr = tr\left\{\dfrac{F_1}{r} - 1\right\}$ . . . . . . . . . . . . . . . . . . . . . . . . . (2)

and $t\rho = \rho_2 P_2 = FR_2 = d$, the distance between the centers of first generator and director . . . . . . . . . . . (3)

If the radii be given for a *minor* hypotrochoid then $FQ : \rho_2 P_2 :: Fq : \rho_2 q$,

from which we have, as before,

$$\textit{fixed radius desired} = \frac{\textit{radius of given fixed circle} \times \textit{given tracing radius}}{\textit{radius of given generator}} \quad \text{. . . . . . . . . . . . . . (4)}$$

and, similarly, formulae (2) and (3) give the radius of desired generator and the corresponding tracing radius.

With the tracing point *on* the circumference of the generator, if we let $R$ = radius of the latter for a *major* hypocycloid and $r$ correspondingly for the *minor* curve, then

for a major hypocycloid $R = F - r$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (5)

" a minor " $r = F - R$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (6)

For the curves intermediate between the major and minor hypotrochoids, viz., those traced when the diameter of the rolling circle is exactly half that of the fixed circle, a separate division seems essential to completeness, and for such I suggest the general name of *medial hypotrochoids.* For these the formulae for double generation are the same as for the "major" and "minor" curves, and similarly derived.

With the tracing point *on* the circumference of the generator these curves reduce to straight lines, diameters of the director. In all other cases the medial hypotrochoids are an interesting exception to what we might naturally expect, being neither looped nor wavy, but *ellipses.* The failure of the terms "looped" and "wavy" to apply to these medial curves is paralleled by that of the adjectives "curtate" and "prolate," since, contrary to the signification of the latter terms, any ellipse generated as a curtate curve is larger than the largest prolate elliptical hypotrochoid having the same director. And as we have seen that, with scarcely an exception, "curtate" and "prolate" apply equally to the same curve, our only reason for retaining them is the fact of their general acceptation as indicative of the location of the tracing point with respect to the circumference of the rolling circle.

Since the medial hypotrochoids are either straight lines or ellipses, we can readily find for them that which we have found it useless to attempt to construct for the other trochoidal curves, viz., simple terms suggestive of their *form;* in fact the names "straight hypocycloid" and "elliptical hypotrochoid" have long been familiar to us all, and we have but to incorporate them into the nomenclature we are constructing.

It only remains to show that a prolate *epi*-trochoid can be generated as a curtate *peri*-trochoid, and vice versa, for which the demonstration is analogous to that given for the hypo-curves and leads to the following formulae, derived from the similar triangles $QFq$ and $QR_1P_1$ (the values being supposed to be given for the *epi*-trochoid and desired for the *peri*-trochoid):

$F_2 = \dfrac{F_1\,(tr)}{r}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (7)

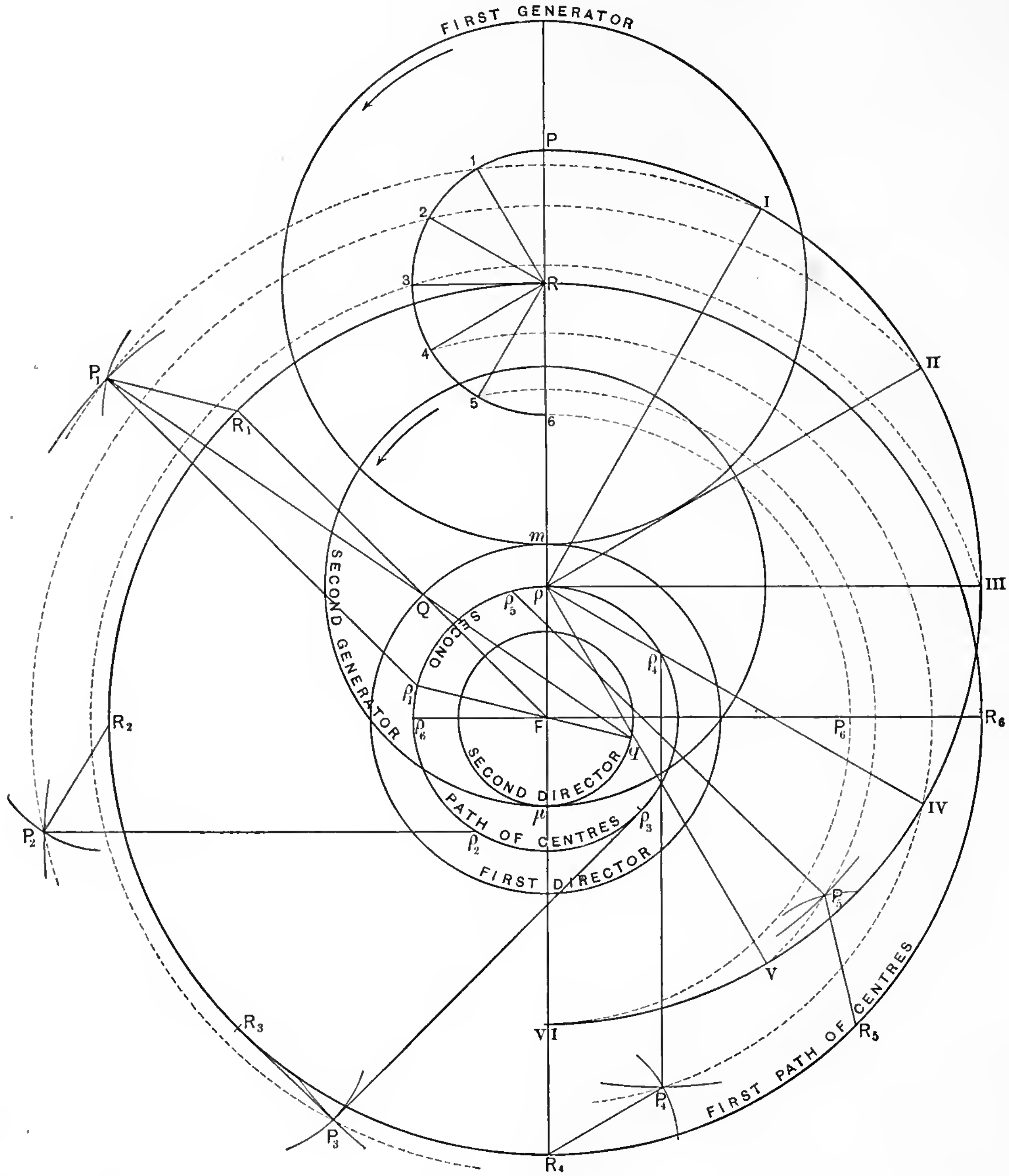
$\rho = tr\left\{\dfrac{F_1}{r} + 1\right\}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (8)

$t\rho = d$ = distance between centres of given generator and director $= F_1 + r$ . . . . . . . . . . . . . . . . (9)

If given as a *peri*-trochoid and desired as an *epi*-trochoid the tracing radius will again equal the distance between the

given centres (in this case, however = R — F); the formula of the radius of desired director will be of the same form

Fig. 5.




as equations (1) and (7); but

radius of second generator $= t\,r \left\{ 1 - \frac{F_1}{r} \right\}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (10)

With the tracing point *on* the circumference of the generator, and letting R = radius of the same for a peritrochoid and $r$ for an epitrochoid, we have

for the epicycloid $r = R - F$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (11)

" " pericycloid $R = F + r$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (12)

ALPHABETS

AND ORNAMENTAL DEVICES

FOR TITLES.

No. 1.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 2.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z 1 2 3 4 5 6 7 8 9 0

No. 3.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
1 2 3 4 5 & 6 7 8 9 0

No. 4.

*Specimens of the modified Italic form called "Reinhardt Gothic," its various forms and applications having been handsomely illustrated by C.W. Reinhardt, in a special text-book devoted almost exclusively to this form. It is much used in engineering, chiefly on account of its compactness, and its legibility after reduction by photo-processes. An inclined ellipse is the basis of many of the letters. The G and S are peculiar, also the Q. Beginners usually make the stems of the p, b, etc., too long. The forms of the numerals should be particularly noted.*

*abcdefghijklmnopqrstuvwxyz. 1234567890*

*CDGPRQJKMMNB & EAVWUTLHSFZYI*

*Use "ball-point" pens for above letters.*

No. 5.

*A B C D E F G H I J K L M N O P Q R S T U V W X Y Z*
*1 2 3 4 5 & 6 7 8 9 0*

No. 6.

A B C D E F G H I J K L M N O P Q R S T U V
W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 7.

A B B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

No. 8.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v w x y z . , 1 2 3 4 5 6 7 8 9 0

No. 9.

A B B C D E F G H I J K L M N N O O P Q R R S
T U V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 10.

A B C D E F G H I J K L M N O P Q R S T U V W X
Y Z & a b c d e f g h i j k l m n o p q r s t u v w
x y z . , 1 2 3 4 5 6 7 8 9 0

No. 11.

A B C D E F G H I J K L M N O P Q R S T U V
W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 12.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
1 2 3 4 5 & 6 7 8 9 0

No. 13.

A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z & a b c d e f g h i j k l m n o
1 2 3 4 5 p q r s t u v w x y z 6 7 8 9 0

No. 14.

A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z & a b c d e f g h i j k l m n o
p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 15.

A B C D E F G H I J K L M N O P Q R S
T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v w x y z
1 2 3 4 5 6 7 8 9 0

No. 16.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
1 2 3 4 5 & 6 7 8 9 0

No. 17.

A B C D E F G H I J K L M N O P Q R S T U
V W X Y Z & a b c d e f g h i j k l m n o p q r s
t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 18.

A B C D E F G H I J L K L M N O P Q R S T U V
W X Y Z & a b c d e f g h i j k l m n o p q r s t
u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 19.

A B C D E F G H I J K L M N O P Q R S T
1 2 3 4 5 U V W X Y Z & 6 7 8 9 0
a b c d e f g h i j k l m n o p q r s t u v w x y z

No. 20.

A B C D E F G H I J K L M N O
P Q R S T U V W X Y Z & a b c
d e f g h i j k l m n o p q r s t u
v w x y z 1 2 3 4 5 6 7 8 9 0

No. 21.

A B C D E F G H I J K L M N O
P Q R S T U V W X Y Z & a b c d
e f g h i j k l m n o p q r s t u v w x y z
1 2 3 4 5 6 7 8 9 0

No. 22.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0 , .

No. 23.

*A B C D E F G H I J K L M N O P Q R S*
*T U V W X Y Z & a b c d e f g h i j k l m n*
*o p q r s t u v w x y z . , 1 2 3 4 5 6 7 8 9 0*

No. 24.

A B C D E F G H I J K L M N O P Q R S T U V
W X Y Z & A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z . , 1 2 3 4 5 6 7 8 9 0

No. 25.

A B C D E F G H I J K L M N O P Q R S T U V
W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 26.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
& 1 2 3 4 5 6 7 8 9 0 , .

No. 27.

A B C D E F G H I J K L M N O P Q R S

T U V W X Y Z & a b c d e f g h ij k l m n

o p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 28.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z &

a b c d e f g h i j k l m n o p q r s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 29.

A B C D E F G H I J K L M N O P Q R S T

U V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 30.

A B C D E F G H I J K L M N O P Q R S T

U V W X Y Z &

a b c d e f g h i j k l m n o p q r s t u v w x y z

1 2 3 4 5 6 7 8 9 0

No. 31.

A B C D E F G H I J K L M N O P Q

R S T U V W X Y Z

No. 32.

A B C D E F G H I J K L M N O P Q R S T

U V W X Y Z &

No. 33.

A B C D E F G H I J K L M N O P Q R S
T U V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 34.

A B C D E F G H I J K L M N O P Q R S T U V
a b c d e f g h i j k l m W X Y Z & n o p q r s t u v w x y z

No. 35.

A B C D E F G H I K L M N O P Q
R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u v
w x y z 1 2 3 4 5 6 7 8 9 0

No. 36.

A B C D E F G H I J K L M N O P
Q R S T U V W X Y Z & a b c d e f
g h i j k l m n o p q r s t u v w x y z
1 2 3 4 5 6 7 8 9 0

No. 37.

A A A B B C D D D E E F G H H H
I J J K L M N N N O O P Q R S T U
V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 38.

A B C D E F G H I J K L M N O P Q R S T
U V W X Y Z & a b c d e f g h i j k l m n
o p q r s t u v w x y z . , 1 2 3 4 5 6 7 8 9 0

No. 39.

A B C D E F G H I J K L M N O P Q R S T U
V W X Y Z & a b c d e f g h i j k l m n o p q r
s t u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 40.

A B C D E F G H I J K L M N
O P Q R S T U V W X Y Z &

No. 41.

A B C D E F G H I J K L
M N O P Q R S T U V W
X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 42.

A B C D E F G H I J K L
M N O P Q R S T U V W
X Y Z & a b c d e f g h i j k l
m n o p q r s t u v w x y z . ,
1 2 3 4 5 6 7 8 9 0

No. 43.

A B C D E F G H I J K L M N O P
Q R S T U V W X Y Z &

No. 44.

A B C D E F G H I J K L M
N O P Q R S T U V W X Y Z
a b c d d e f f g h h i j k l m n n o p q r s t u v w x y z
1 2 2 3 4 5 6 7 7 8 9 0

(The letters above are the original Soennecken forms, used by permission of Messrs. Keuffel & Esser, New York, holders of the American copyright and agents for the special pens and copy-books required).

No. 45.

A B C D E F G H I J K L M N O P Q R S
T U V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 46.

A B C D E F G H I J K L M N O
P Q R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t
u v w x y z 1 2 3 4 5 6 7 8 9 0

No. 47.

A B C D E F G H I J K L M N O P
Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

No. 48.

A B C D E F G H I J K L M N
O P Q R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s
1 2 3 4 5 t u v w x y z 6 7 8 9 0

No. 49.

A B C D E F G H I J K L M N O P
Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

No. .50

A B C D E F G H I J K L M N O P
Q R S T U V W X Y Z &
1 2 3 4 5 6 7 8 9 0

No. 58.

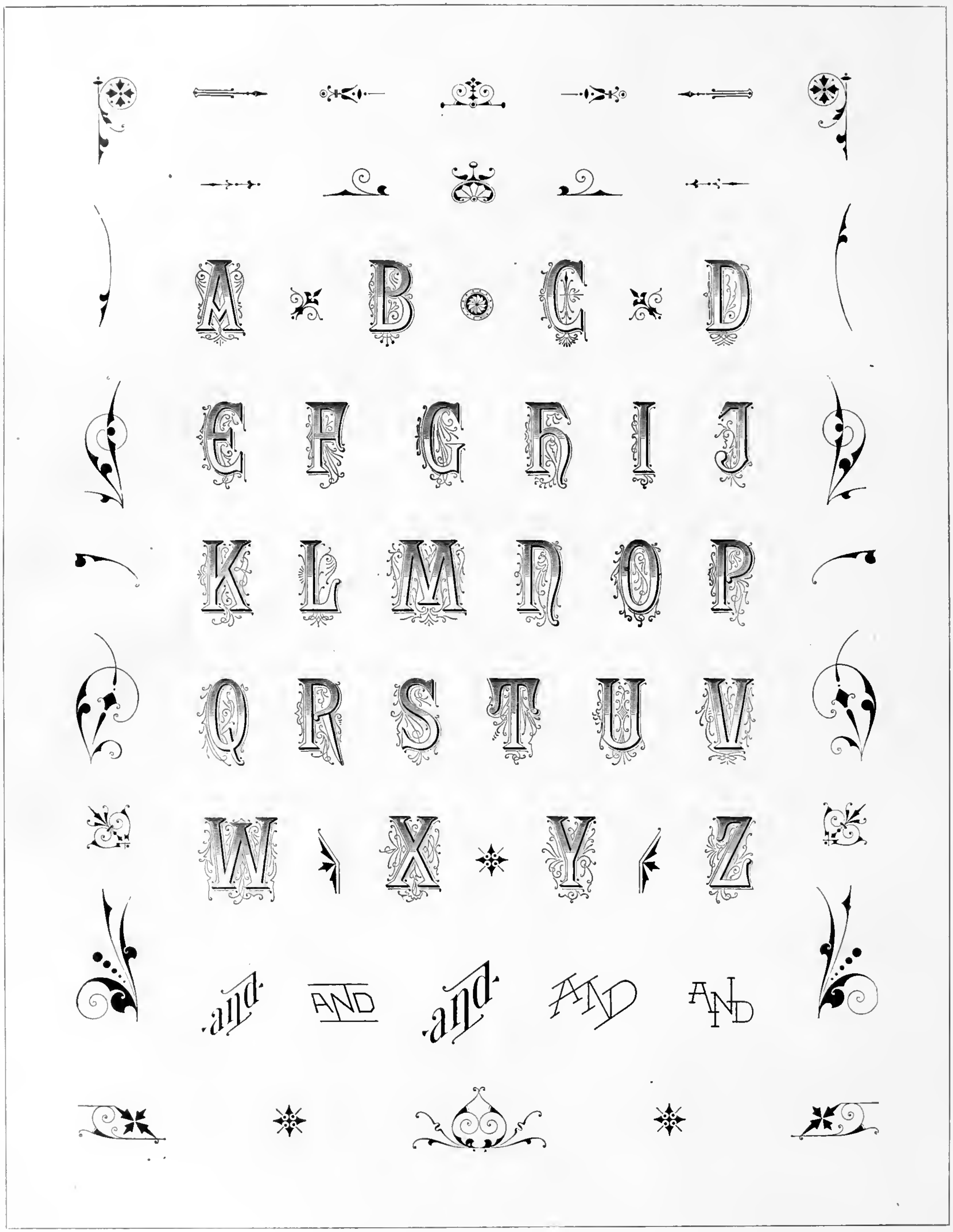
A B C D E F G H I J K L M N O P Q R S T U
V W X Y Z & a b c d e f g h i j k l m n o p q r
s t u v w x y z . , 1 2 3 4 5 6 7 8 9 0

No. 59.

A B C D E F G H I J K L M N O P Q R
S T U V W X Y Z & 1 2 3 4 5 6 7 8 9 0

No. 60.

A B C D E F G H I J K L M N
O P Q R S T U V W X Y Z &
a b c d e f g h i j k l m n o p q r s t u
v w x y z 1 2 3 4 5 6 7 8 9 0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# INDEX

# AND LIST OF

# REFERENCE WORKS.

# INDEX.

# INDEX.

# INDEX.

# INDEX.

## *INDEX.*



## VALUABLE REFERENCE LITERATURE ON GRAPHICAL SCIENCE.

### LINEAR AND MACHINE DRAWING.

The Engineer and Machinist's Drawing Book; Delaistre, *Cours de Dessin Linéaire;* Appleton's *Cyclopædia of Drawing;* Ripper's *Machine Drawing and Design;* Clarke's *Practical Geometry and Engineering Drawing;* Low and Bevis, *Manual of Machine Drawing and Design.*

### PLANE CURVES.

Leslie, *Geometrical Analysis;* Salmon, *Higher Plane Curves;* Eagles, *Constructive Geometry of Plane Curves;* Proctor, *Geometry of Cycloids;* Burmester, *Lehrbuch der Kinematik:* also chapters in the Descriptive Geometries of Peschka and Wiener.

### KINEMATICS AND MECHANISM.

Reuleaux, *Kinematics of Machinery;* Willis, *Principles of Mechanism;* Clifford, *Kinematic:* Weisbach, *Machinery of Transmission;* Kempe, *How to Draw a Straight Line:* Burmester, *Lehrbuch der Kinematik;* Grant, *Odontics;* Goodeve, *Principles of Mechanism.*

### PROJECTIVE GEOMETRY.

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### MONGE'S DESCRIPTIVE GEOMETRY.

Authors: Monge, Lacroix, Hachette, Vallée, Lefébure de Fourcy, Leroy, Olivier, De la Gournerie, Adhémar, Mannheim, Songaylo, Javary, Caron, Fiedler, Peschka, Wiener, Marx, Woolley, Eagles, Pierce, Church, Warren.

### SHADES AND SHADOWS.

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### PERSPECTIVE.

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### AXONOMETRIC AND OBLIQUE PROJECTION.

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### ONE-PLANE DESCRIPTIVE GEOMETRY.

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